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STUDY GUIDE

# MATHEMATICS HL

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# IB Academy Mathematics Study Guide

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# INTRODUCTION

Welcome to the IB.Academy Study Guide for IB Mathematics High Level.

We are proud to present our study guides and hope that you will find them helpful. They are the result of a collaborative undertaking between our tutors, students and teachers from schools across the globe. Our mission is to create the most simple yet comprehensive guides accessible to IB students and teachers worldwide. We are firm believers in the open education movement, which advocates for transparency and accessibility of academic material. As a result, we embarked on this journey to create these study guides that will be continuously reviewed and improved. Should you have any comments, feel free to contact us.

For this Mathematics HL guide, we incorporated everything you need to know for your final exam. The guide is broken down into chapters based on the syllabus topics and they begin with 'cheat sheets' that summarise the content. This will prove especially useful when you work on the exercises. The guide then looks into the subtopics for each chapter, followed by our step-by-step approach and a calculator section which explains how to use the instrument for your exam.

For more information and details on our revision courses, be sure to visit our website at ib.academy. We hope that you will enjoy our guides and best of luck with your studies.

IB.Academy Team

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**Arithmetic: +/− common difference**

$$u_n = n^{\text{th}} \text{ term} = u_1 + (n-1)d$$
$$S_n = \text{sum of } n \text{ terms} = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with $u_1 = a = 1^{\text{st}}$ term, $d =$ common difference.

**Geometric: ×/÷ common ratio**

$$u_n = n^{\text{th}} \text{ term} = u_1 \cdot r^{n-1}$$
$$S_n = \text{sum of } n \text{ terms} = \frac{u_1(1-r^n)}{(1-r)}$$
$$S_\infty = \text{sum to infinity} = \frac{u_1}{1-r}, \text{ when } -1 < r < 1$$

with $u_1 = a = 1^{\text{st}}$ term, $r =$ common ratio.

**Sigma notation**
A shorthand to show the sum of a number of terms in a sequence.

Last value of $n$

$$\sum_{n=1}^{10} 3n-1 \longleftarrow \text{Formula}$$

First value of $n$

e.g.

$$\sum_{n=1}^{10} 3n-1 = \underbrace{(3\cdot 1)-1}_{n=1} + \underbrace{(3\cdot 2)-1}_{n=2} + \cdots + \underbrace{(3\cdot 10)-1}_{n=10} = 155$$

### 1.2. Exponents and logarithms 12

**Exponents**

$$x^1 = x \qquad x^0 = 1$$
$$x^m \cdot x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$
$$(x^m)^n = x^{m\cdot n} \qquad (x\cdot y)^n = x^n \cdot y^n$$
$$x^{-1} = \frac{1}{x} \qquad x^{-n} = \frac{1}{x^n}$$
$$x^{\frac{1}{2}} = \sqrt{x} \qquad \sqrt{x}\cdot\sqrt{x} = x$$
$$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y} \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$
$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad x^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{x^m}}$$

**Logarithms**

$$\log_a a^x = x \qquad a^{\log_a b} = b$$

*Let* $a^x = b$*, isolate x from the exponent:* $\log_a a^x = x = \log_a b$

*Let* $\log_a x = b$*, isolate x from the logarithm:* $a^{\log_a x} = x = a^b$

**Laws of logarithms**

**I:** $\log A + \log B = \log(A\cdot B)$

**II:** $\log A - \log B = \log\left(\frac{A}{B}\right)$

**III:** $n\log A = \log(A^n)$

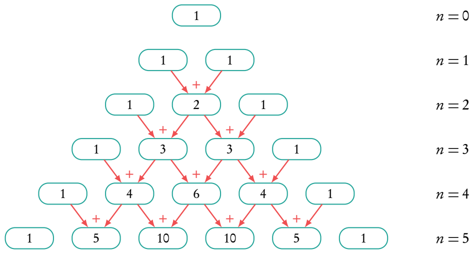
**IV:** $\log_B A = \frac{\log A}{\log B}$

### 1.3. Binomial Expansion 14

In a expansion of a binomial in the form $(a+b)^n$. Each term can be described as $\binom{n}{r}a^n - r b^r$, where $\binom{n}{r}$ is the coefficient.
The full expansion can be written thus

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Find the coefficient using either pascals triangle



Or the nCr function on your calculator

# 1.1 Sequences

## 1.1.1 Arithmetic sequence

**Arithmetic sequence** the next term is the previous number + the common difference $(d)$.

To find the common difference $d$, subtract two consecutive terms of an arithmetic sequence from the term that follows it, i.e. $u_{(n+1)} - u_n$.

DB 1.1 Use the following equations to calculate the $n^{\text{th}}$ term or the sum of $n$ terms.

$$u_n = u_1 + (n-1)d \qquad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad d = \text{common difference}$$

Often the IB requires you to first find the $1^{\text{st}}$ term and/or common difference.

**Finding the first term $u_1$ and the common difference $d$ from other terms.**

In an arithmetic sequence $u_{10} = 37$ and $u_{22} = 1$. Find the common difference and the first term.

| | | |
|---|---|---|
| **1.** | Put numbers in to $n^{\text{th}}$ term formula | $37 = u_1 + 9d$<br>$1 = u_1 + 21d$ |
| **2.** | Equate formulas to find $d$ | $21d - 1 = 9d - 37$<br>$12d = -36$<br>$d = -3$ |
| **3.** | Use $d$ to find $u_1$ | $1 - 21 \cdot (-3) = u_1$<br>$u_1 = 64$ |

### 1.1.2 Geometric sequence

**Geometric sequence** the next term is the previous number multiplied by the common ratio ($r$).

To find the common ratio, divide any term of an arithmetic sequence by the term that precedes it, i.e. $\frac{\text{second term } (u_2)}{\text{first term } (u_1)}$

Use the following equations to calculate the $n^{\text{th}}$ term, the sum of $n$ terms or the sum to infinity when $-1 < r < 1$.

DB 1.1

$$u_n = n^{\text{th}} \text{ term} \qquad S_n = \text{sum of } n \text{ terms} \qquad S_\infty = \text{sum to infinity}$$

$$= u_1 \cdot r^{n-1} \qquad = \frac{u_1(1-r^n)}{(1-r)} \qquad = \frac{u_1}{1-r}$$

again with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad r = \text{common ratio}$$

Similar to questions on Arithmetic sequences, you are often required to find the 1st term and/or common ratio first.

### 1.1.3 Sigma notation

Sigma notation is a way to represent the summation of any sequence — this means that it can be used for both arithmetic or geometric series. The notation shows you the formula that generates terms of a sequence and the upper and lower limits of the terms that you want to add up in this sequence.

$$\overset{\text{Last value of } n}{\sum_{n=1}^{10}} 3n-1 \longleftarrow \text{Formula}$$

First value of $n$

e.g. $\sum_{n=1}^{10} 3n-1 = \underbrace{(3\cdot 1)-1}_{n=1} + \underbrace{(3\cdot 2)-1}_{n=2} + \underbrace{(3\cdot 3)-1}_{n=3} + \cdots + \underbrace{(3\cdot 10)-1}_{n=10} = 155$

## Finding the first term $u_1$ and common ratio $r$ from other terms.

$\sum_{1}^{5}$(Geometric series) $= 3798$, $\sum_{1}^{\infty}$(Geometric series) $= 4374$.

Find $\sum_{1}^{7}$(Geometric series) $=?$

| | | |
|---|---|---|
| **1.** | Interpret the question | The sum of the first $5$ terms of a geometric sequence is $3798$ and the sum to infinity is $4374$. Find the sum of the first $7$ terms |
| **2.** | Use formula for sum of $n$ terms | $3798 = u_1 \frac{1-r^5}{1-r}$ |
| **3.** | Use formula for sum to infinity | $4374 = \frac{u_1}{1-r}$ |
| **4.** | Rearrange **3.** for $u_1$ | $4374(1-r) = u_1$ |
| **5.** | Substitute in to **2.** | $3798 = \frac{4374(1-r)\left(1-r^5\right)}{1-r}$ |
| **6.** | Solve for $r$ | $3798 = 4374\left(1-r^5\right)$<br>$\frac{3798}{4374} = 1-r^5$<br>$r^5 = 1-\frac{211}{243}$<br>$\sqrt[5]{r} = \sqrt[5]{\frac{32}{243}}$<br>$r = \frac{2}{3}$ |
| **7.** | Use $r$ to find $u_1$ | $u_1 = 4374\left(1-\frac{2}{3}\right)$<br>$u_1 = 1458$ |
| **8.** | Find sum of first 7 terms | $1458\frac{1-\left(\frac{2}{3}\right)^7}{1-\frac{2}{3}} = 4370$ |

## 1.1.4 The fundamental theorem of algebra and complex roots

**The fundamental theorem of algebra** any polynomial of degree $n$ has $n$ roots

A degree of a polynomial is the largest exponent.

**Example.**

If $f(x) = 4x^3 + 3x^2 + 7x + 9$ then it is a polynomial at degree 3, and according to the fundamental theorem of algebra will have 3 roots.

Any polynomial can be rewritten/factorized to include the roots:

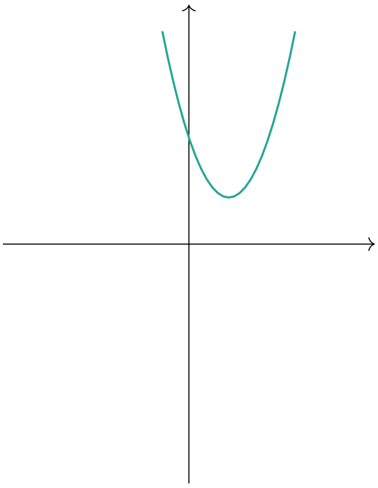
$$a(x - r_1)(x - r_2)(x - r_3)\cdots$$

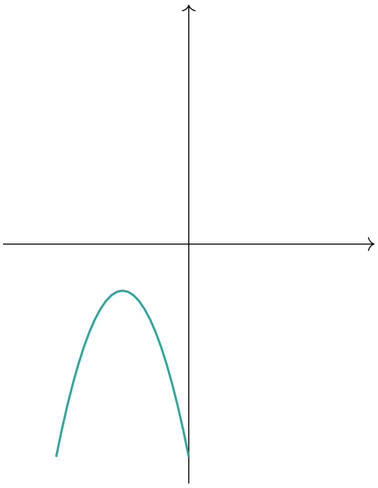
where $r_1, r_2, r_3, \ldots,$ are all roots.

Note: some polynomials will have complex roots. A polynomial of degree 4 can have 4 real roots or 4 complex roots or 2 real and 2 complex roots.

**Example.**

**Complex roots of quadratic equations always come in conjugate pairs.**
For example: $2x^2 - 3x + 4$ has complex roots, i.e. $(b^2 - 4ac) < 0$. The roots are $x = \frac{3}{4} + \frac{i\sqrt{7}}{4}$ and $x = \frac{3}{4} - \frac{i\sqrt{7}}{4}$ can be shown graphically.





# 1.2 Exponents and logarithms

## 1.2.1 Laws of exponents

Exponents always follow certain rules. If you are multiplying or dividing, use the following rules to determine what happens with the powers.

**Example.**

$$x^1 = x \qquad 6^1 = 6$$

$$x^0 = 1 \qquad 7^0 = 1$$

$$x^m \cdot x^n = x^{m+n} \qquad 4^5 \cdot 4^6 = 4^{11}$$

$$\frac{x^m}{x^n} = x^{m-n} \qquad \frac{3^5}{3^4} = 3^{5-4} = 3^1 = 3$$

$$(x^m)^n = x^{m \cdot n} \qquad \left(10^5\right)^2 = 10^{10}$$

$$(x \cdot y)^n = x^n \cdot y^n \qquad (2 \cdot 4)^3 = 2^3 \cdot 4^3 \quad \text{and} \quad (3x)^4 = 3^4 x^4$$

$$x^{-1} = \frac{1}{x} \qquad 5^{-1} = \frac{1}{5} \quad \text{and} \quad \left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$$

$$x^{-n} = \frac{1}{x^n} \qquad 3^{-5} = \frac{1}{3^5} = \frac{1}{243}$$

## 1.2.2 Fractional exponents

When doing mathematical operations (+, −, × or ÷) with fractions in the exponent you will need the following rules. These are often helpful when writing your answers in simplest terms.

**Example.**

$$x^{\frac{1}{2}} = \sqrt{x} \qquad 2^{\frac{1}{2}} = \sqrt{2}$$

$$\sqrt{x} \cdot \sqrt{x} = x \qquad \sqrt{3} \cdot \sqrt{3} = 3$$

$$\sqrt{xy} = \sqrt{x} \cdot \sqrt{y} \qquad \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2 \cdot \sqrt{3}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x} \qquad 5^{\frac{1}{3}} = \sqrt[3]{5}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad 3^{-\frac{2}{5}} = \frac{1}{\sqrt[5]{3^2}}$$

## 1.2.3 Laws of logarithms

DB 1.2

Logarithms are the inverse mathematical operation of exponents, like division is the inverse mathematical operation of multiplication. The logarithm is often used to find the variable in an exponent.

$$a^x = b \quad \Leftrightarrow \quad x = \log_a b$$

Since $\log_a a^x = x$, so that $x = \log_a b$.

This formula shows that the variable $x$ in the power of the exponent becomes the subject of your log equation, while the number $a$ becomes the base of your logarithm.

Below are the rules that you will need to use when performing calculations with logarithms and when simplifying them. The sets of equations on the left and right are the same; on the right we show the notation that the DB uses while the equations on the left are easier to understand.

### Laws of logarithms and change of base

DB 1.2

| | | |
|---|---|---|
| **I:** | $\log A + \log B = \log(A \cdot B)$ | $\log_c a + \log_c b = \log_c(ab)$ |
| **II:** | $\log A - \log B = \log\left(\frac{A}{B}\right)$ | $\log_c a - \log_c b = \log_c\left(\frac{a}{b}\right)$ |
| **III:** | $n \log A = \log(A^n)$ | $n \log_c a = \log_c(a^n)$ |
| **IV:** | $\log_B A = \frac{\log A}{\log B}$ | $\log_b a = \frac{\log_c a}{\log_c b}$ |

### Note

- $x = \log_a a = 1$
- With the 4th rule you can change the base of a log.
- $\log_a 0 = x$ is always undefined (because $a^x \neq 0$).
- When you see a log with no base, it is referring to a logarithm with a base of 10 (e.g. $\log 13 = \log_{10} 13$).

**Solve $x$ in exponents using logs.**

Solve $2^x = 13$.

| | | |
|---|---|---|
| **1.** | Take the log on both sides | $\log 2^x = \log 13$ |
| **2.** | Use rule III to take $x$ outside | $x \log 2 = \log 13$ |
| **3.** | Solve | $x = \frac{\log 13}{\log 2}$ |

But what about ln and e? These work exactly the same; e is *just* the irrational number 2.71828… (infinitely too long to write out) and ln is just $\log_e$.

$$\begin{aligned}\ln a + \ln b &= \ln(a \cdot b)\\ \ln a - \ln b &= \ln\left(\frac{a}{b}\right)\\ n \ln a &= \ln a^n\\ \ln \mathrm{e} &= 1\\ \mathrm{e}^{\ln a} &= a\end{aligned}$$

## 1.3 Binomial expansion

**Binomial** an expression $(a+b)^n$ which is the sum of *two terms* raised to the power $n$.

**Binomial expansion** $(a+b)^n$ is expanded into a sum of terms

Binomial expansions get increasingly complex as the power increases:

| **binomial** | **binomial expansion** |
|---|---|
| $(a+b)^1$ | $= a+b$ |
| $(a+b)^2$ | $= a^2+2ab+b^2$ |
| $(a+b)^3$ | $= a^3+3a^2b+3ab^2+b^3$ |

The general formula for each term is: $\binom{n}{r}a^{n-r}b^r$.

In order to find the full binomial expansion of a binomial, you have to determine *the coefficient* $\binom{n}{r}$ and *the powers* for each term, $n-r$ and $r$ for $a$ and $b$ respectively, as shown by the binomial expansion formula.

### Binomial expansion formula

DB 1.3

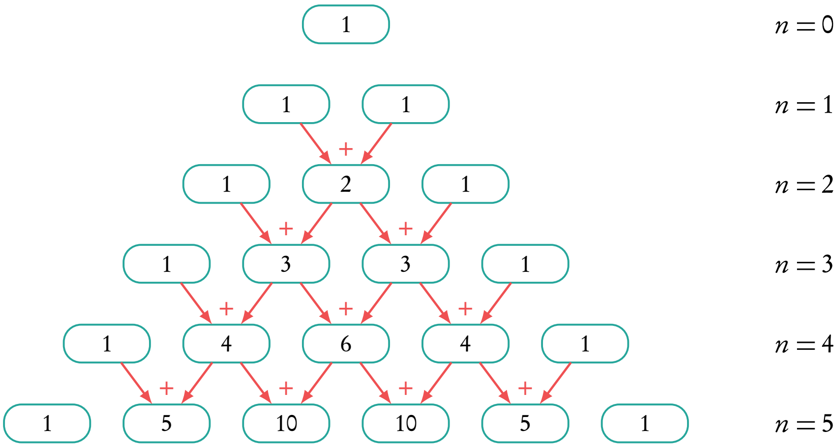
$$\begin{aligned}(a+b)^n = {} & a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n\\ = {} & \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots\end{aligned}$$

The powers decrease by 1 for $a$ and increase by 1 for $b$ for each subsequent term.

The sum of the powers of each term will always $= n$.

There are two ways to find the coefficients: with Pascal's triangle or the binomial coefficient function (nCr).

## Pascal's triangle



*Pascal's triangle* is an easy way to find *all* the coefficients for your binomial expansion. It is particularly useful in cases where:

1. the power is not too high (because you have to write it out manually);
2. if you have to find all the terms in a binomial expansion.

## Binomial coefficient functions

**Combinations** order is not important

$$C_r^n = \frac{n!}{(n-r)!r!} = \binom{n}{r}, \text{ nCr on GDC}$$

**Permutations** order is important

$$P_r^n = \frac{n!}{(n-r)!} = \text{number of ways of choosing } r \text{ objects out of } n = \text{nPr}$$
on GDC

Common types:

1. Arranging in a row
2. Arranging in a circle
3. Arranging letters
4. Arranging numbers

## How to expand binomial expansions.

Find the expansion of $\left(x-\frac{2}{x}\right)^5$

**1.** Use the binomial expansion formula

$a = x$

$b = -\frac{2}{x}$

$$(x)^5+(5C1)(x)^4\left(-\frac{2}{x}\right)+(5C2)(x)^3\left(-\frac{2}{x}\right)^2+(5C3)(x)^2\left(-\frac{2}{x}\right)^3+(5C4)(x)\left(-\frac{2}{x}\right)^4+(5C5)\left(-\frac{2}{x}\right)^5$$

**2.** Find coefficients

*using Pascal's triangle for low powers and nCr calculator for high functions*

Row 0: 1

Row 1: 1 1

Row 2: 1 2 1

Row 3: 1 3 3 1

Row 4: 1 4 6 4 1

$\boxed{\text{Row 5: } 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1}$

$(5C0)=1$ $(5C1)=5$ $(5C2)=10$ $(5C3)=10$ $(5C4)=5$ $(5C5)=1$

**3.** Put them together

$$x^5+5x^4\left(-\frac{2}{x}\right)^1+10x^3\left(-\frac{2}{x}\right)^2+10x^2\left(-\frac{2}{x}\right)^3+5x\left(-\frac{2}{x}\right)^4+\left(-\frac{2}{x}\right)^5$$

**4.** Simplify

*using laws of exponents*

$$x^5-10x^3+20x-\frac{80}{x}+\frac{80}{x^3}-\frac{32}{x^5}$$

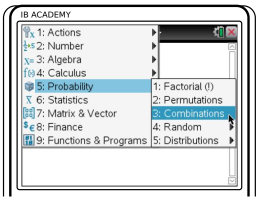
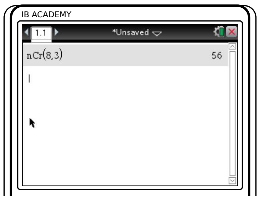
The IB use three different terms for these types of question which will effect the answer you should give:

**Coefficient:** the number before the $x$ value;

**Term:** the number and the $x$ value;

**Constant term:** the number for which there is no $x$ value ($x^0$).

### Finding a specific term in a binomial expansion.

| | | |
|---|---|---|
| | Find the coefficient of $x^5$ in the expansion $(2x-5)^8$ | |
| **1.** | One term is asked, usually of a high power then use binomial expansion formula | $(a+b)^n = \dots + \binom{n}{r} a^{n-r} b^r + \dots$ |
| **2.** | Determine $r$ | Since $a = 2x$, to find $x^5$ we need $a^5$.<br>$a^5 = a^{n-r} = a^{8-r}$, so that $r = 3$ |
| **3.** | Plug $r$ into the general formula | $\binom{n}{r} a^{n-r} b^r = \binom{8}{3} a^{8-3} b^3 = \binom{8}{3} a^5 b^3$ |
| **4.** | Replace $a$ and $b$ | $\binom{8}{3} (2x)^5 (-5)^3$ |
| **5.** | Use nCr to calculate the value for $\binom{n}{r}$ | $\binom{8}{3} = 8C3 = 56$ |
| |  <br>Press menu<br>`5: Probability`<br>`3: Combinations` |  <br>nCr(8,3) 56<br>Insert the values for $n$ and $r$ separated by a comma |
| **6.** | Substitute and calculate the value | $56 \times 2^5(x^5) \times (-5)^3 = -224000(x^5)$ |
| **7.** | Alternatively can be found using: | $\left(\frac{8!}{5!3!}\right) = \left(\frac{8 \times 7 \times 6}{6} = 8 \times 7\right)$ |

## 1.4 Induction

Unlike direct proofs, where the result follows as a logical step, mathematical induction is a form of indirect proof (the only covered in the IB syllabus). Indirect proofs tend to require a ‘creative’ step, however through training one can familiarise most forms of induction.

Prrof by induction always can be split up into three components, that together prove the wanted statement:

① → Showing the statement holds for the first case, $n = 1$

② → Assuming the statement holds true for some value, $n = k$

③ → Proving, using the assumption in 2, that the statement holds for $n = k + 1$

Common types: $f(n) > g(n)$, $f(n) = g(n)$, $f(n) < g(n)$, $\Sigma$, $P(n)$ is a multiple / divisible.

### Induction

Use induction to prove that $5 \times 7^n + 1$ is divisible by 6, $n \in \mathbb{Z}^+$

| | | |
|---|---|---|
| **1.** | Write statement in math form | $P(n) = 5 \times 7^n + 1 = 6A$,<br>$A \in \mathbb{N}$ |
| **2.** | Check first case | $P(1) = 5 \times 7^1 + 1 = 36$,<br>which is divisible by 6 |
| **3.** | Assume $P(n)$ is true for $n = k$ | $5 \times \left(7^k\right) - 1 = 6A \Rightarrow 7^k = \dfrac{6A+1}{5}$ |
| **4.** | Show $P(k+1)$ is true. | $5 \times \left(7^{(k+1)}\right) - 1 = 6B$<br>using assumption:<br>$5 \times 7 \times \left(\dfrac{6A+1}{5}\right) - 1 = 6B$<br>$42A + 7 = 6B + 1$<br>$6(7A+1) = 6B$ |
| **5.** | Write concluding sentence | Hence, since $P(1)$ true and assuming $P(k)$ true, we have shown by the principle of mathematical induction that $P(k+1)$ true. Therefore, $5 \times (7^n) - 1$ is divisible by 6 for all positive integers. |

## 1.5 Complex numbers

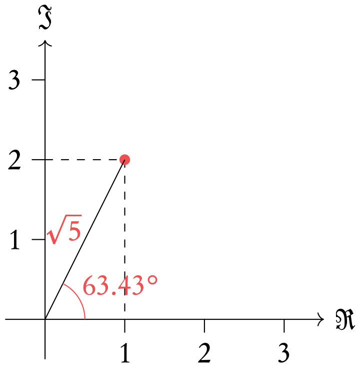
**A complex number** is defined as $z = a + b\mathrm{i}$. Where $a, b \in \mathbb{R}$, $a$ is the real part ($\Re$) and $b$ is the imaginary part ($\Im$).

$$\mathrm{i} = \sqrt{-1}$$
$$\mathrm{i}^2 = -1$$

$z = a + b\mathrm{i}$ is the Cartesian form. $z = r(\cos\theta + \mathrm{i}\sin\theta)$ is the polar form where $r$ is the modulus and $\theta$ is the argument also sometimes stated as $z = r\operatorname{cis}\theta$.

**Modulus $r$** the absolute distance from the origin to the point.

**Argument $\theta$** the angle between the $x$-axis and the line connecting the origin and the point.



Instead of working in $(x, y)$ coordinates, polar coordinates use the distance from the origin to the point ($r$, modulus) and the angle between the $x$-axis and the modulus (argument).
$2 = 1 + 2\mathrm{i} \Rightarrow r = \sqrt{1^2 + 2^2} = \sqrt{5}$
and $\theta = \arctan(2) = 63.43°$
and $\sqrt{5} \times \sin(63.43) = 2$,
$\sqrt{5} \times \cos(63.43) = 1$

**The conjugate of a complex number** $\overline{z}$ or $z^*$, is defined as
$z = a + b\mathrm{i} \Rightarrow \overline{z} = a - b\mathrm{i}$

## De Moivre’s theorem

DB

$$z^n = (\cos x + \mathrm{i}\sin x)^n = \cos(nx) + \mathrm{i}\sin(nx)$$

## Euler’s formula

$$\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x \qquad \text{and} \qquad (\mathrm{e}^{\mathrm{i}x})^n = \mathrm{e}^{\mathrm{i}nx}$$

## De Moivre’s theorem: proof by induction

Having seen the method of induction, we will now apply it to De Moivre’s theorem.

Prove: $z^n = \big(\cos(x)\mathrm{i}\sin(x)\big)^n = \cos(nx) + \mathrm{i}\sin(nx)$.

1. Show true for $n = 1$

$$\big[\cos(x)\mathrm{i}\sin(x)\big]^1 = \cos(1x) + \mathrm{i}\sin(1x)$$
$$\cos(x)\mathrm{i}\sin(x) = \cos(x) + \mathrm{i}\sin(1x)$$

   is true for $n = 1$

2. Assume true for $n = k$

$$\big[\cos(x) + \mathrm{i}\sin(x)\big]^k = \cos(kx) + \mathrm{i}\sin(kx)$$

3. Prove true for $n = k + 1$

$$\begin{aligned}\big[\cos(x) + \mathrm{i}\sin(x)\big]^{k+1} &= \cos\big((k+1)x\big) + \mathrm{i}\sin\big((k+1)x\big)\\ &= \big(\cos(x) + \mathrm{i}\sin(x)\big)^1\big(\cos(x) + \mathrm{i}\sin(x)\big)^k\end{aligned}$$

   Inductive step: use assumption about $n = k$

$$= \big(\cos(x) + \mathrm{i}\sin(x)\big)\big(\cos(kx) + \mathrm{i}\sin(kx)\big)$$

   Remember $\mathrm{i}^2 = -1$

$$= \big(\cos(x)\big)\big(\cos(kx)\big) + \big(\cos(x)\big)\big(\mathrm{i}\sin(kx)\big) + \big(\mathrm{i}\sin(x)\big)\big(\cos(kx)\big) - \big(\sin(x)\big)\big(\sin(kx)\big)$$

$$= \cos(x)\cos(kx) - \sin(x)\sin(kx) + \mathrm{i}\big(\cos(x)\sin(kx) + \sin(x)\cos(kx)\big)$$

   Use of the double/half angle formulae

$$\begin{aligned}&= \cos\big(\theta + k\theta\big) + \mathrm{i}\sin\big(\theta + k\theta\big)\\ &= \cos\big((k+1)\theta\big) + \mathrm{i}\sin\big((k+1)\theta\big)\end{aligned}$$

   is required result and form.

Hence, by assuming $n = k$ true, $n = k + 1$ is true. Since the statement is true for $n = 1$, it is true for all $n \in \mathbb{Z}^+$.

# FUNCTIONS

2

## Table of contents & cheatsheet

### Definitions

| | |
|---|---|
| Function | a mathematical relationship where each input has a single output. It is often written as $f(x)$ where $x$ is the input |
| Domain | all possible $x$ values, the input. (the domain of investigation) |
| Range | possible $y$ values, the output. (the range of outcomes) |
| Coordinates | uniquely determines the position of a point, given by $(x,y)$ |

### 2.1. Types of functions 22

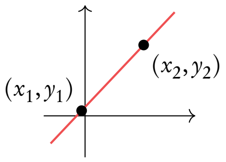
**Linear functions** $y = mx + c$

$m$ is the *gradient*, $c$ is the $y$ *intercept*.

**Midpoint:** $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

**Distance:** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

**Gradient:** $m = \dfrac{y_2-y_1}{x_2-x_1}$

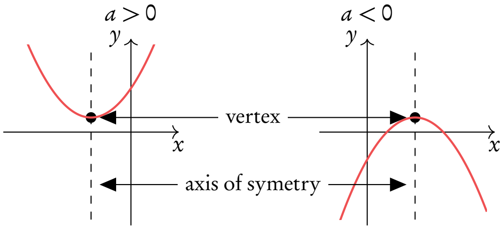


**Parallel lines:** $m_1 = m_2$ (same gradients)
**Perpendicular lines:** $m_1 m_2 = -1$

**Quadratic functions** $y = ax^2 + bx + c = 0$

**Axis of symmetry:** $x$-coordinate of the vertex: $x = \dfrac{-b}{2a}$

**Factorized form:** $y = (x+p)(x+q)$

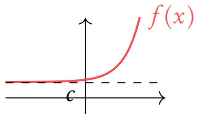


**If** $a = 1$ use the factorization method $(x+p)\cdot(x+q)$
**If** $a \neq 1$ use the quadratic formula
**When asked excplicity** complete the square

**Vertex form:** $y = a(x-h)^2 + k$
**Vertex:** $(h,k)$

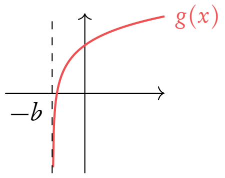
**Exponential**

$f(x) = a^x + c$



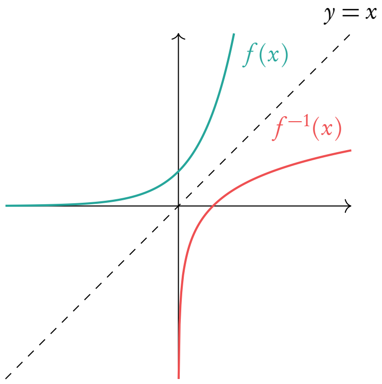
**Logarithmic**

$g(x) = \log_a(x+b)$



### 2.2. Rearranging functions 28

**Inverse function, $f^{-1}(x)$** reflection of $f(x)$ in $y = x$.



**Composite function, $(f \circ g)(x)$** is the combined function $f$ of $g$ of $x$.

When $f(x)$ and $g(x)$ are given, replace $x$ in $f(x)$ by $g(x)$.

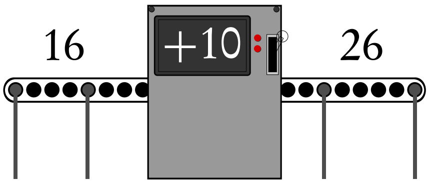
**Transforming functions**

| Change to $f(x)$ | Effect |
|---|---|
| $f(x)+a$ | Move graph $a$ units upwards |
| $f(x+a)$ | Move graph $a$ units to the left |
| $a\cdot f(x)$ | Vertical stretch by factor $a$ |
| $f(a\cdot x)$ | Horizontal stretch by factor $\frac{1}{a}$ |
| $-f(x)$ | Reflection in $x$-axis |
| $f(-x)$ | Reflection in $y$-axis |

ACADEMY

## 2.1 Types of functions

Functions are mathematical relationships where each input has a single output. You have probably been doing functions since you began learning maths, but they may have looked like this:



Algebraically this is:
$f(x) = x + 10$,
here $x = 16$, $y = 26$.

We can use graphs to show multiple outputs of $y$ for inputs $x$, and therefore visualize the relation between the two. Two common types of functions are linear functions and quadratic functions.

### 2.1.1 Linear functions

**Linear functions** $y = mx + c$ increases/decreases at a constant rate $m$, where $m$ is the *gradient* and $c$ is the $y$ *intercept*.

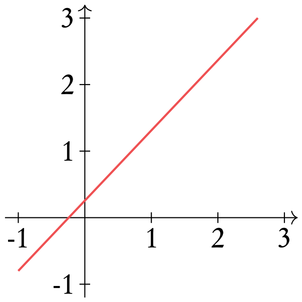
**Midpoint** $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

**Distance** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

**Gradient** $m = \frac{y_2-y_1}{x_2-x_1}$

**Parallel lines** $m_1 = m_2$ (equal gradients)

**Perpendicular lines** $m_1 m_2 = -1$



Example.

**Determine the midpoint, distance and gradient using the two points $P_1(2,8)$ and $P_2(6,3)$**

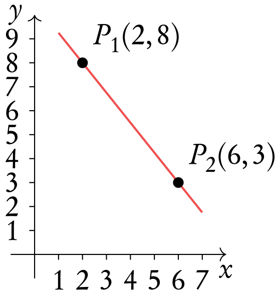
**Midpoint:** $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{2+6}{2}, \frac{8+3}{2}\right) = (4, 5.5)$

**Distance:** $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(6-2)^2+(3-8)^2} = \sqrt{(4)^2+(5)^2} = \sqrt{41}$

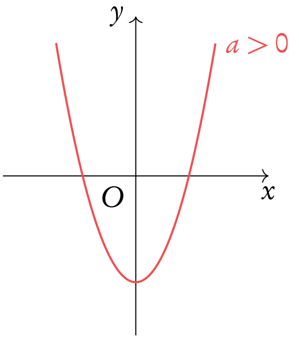
**Gradient:** $m = \frac{y_2-y_1}{x_2-x_1} = m = \frac{3-8}{6-2} = -\frac{5}{4}$

**Parallel line:** $-\frac{5}{4}x + 3$

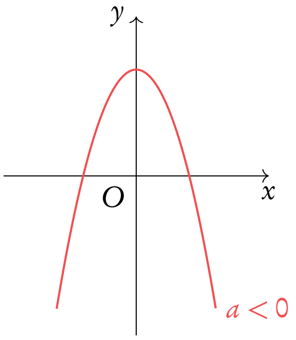
**Perpendicular line:** $-\frac{4}{5}x + 7$



## 2.1.2 Quadratic functions



$a > 0$, positive quadratic



$a < 0$, negative quadr

**Quadratic functions** $y = ax^2 + bx + c = 0$

Graph has a parabolic shape, increase/decrease at an increasing rate.

The roots of an equation are the $x$-values for which $y = 0$, in other words the $x$-intercept(s).

To find the roots of the equation you can use

**factorisation:** If $a = 1$, use the factorization method $(x+p)\cdot(x+q)$

**quadratic formula:** If $a \neq 1$, use the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

The $b^2 - 4ac$ part of the quadratic formula is also known as the discriminant $\Delta$. It can be used to check how many $x$-intercepts the equation has:
$\Delta > 0$: 2 solutions
$\Delta = 0$: 1 solution
$\Delta < 0$: no real solutions

**Solving quadratic equations by factorisation.**

Solve: $x^2 - 5x + 6 = 0$

| | | |
|---|---|---|
| **1.** | Set up system of equations<br>$p + q = b$ and $p \times q = c$ | $\left.\begin{aligned} p + q &= -5 \\ p \times q &= 6 \end{aligned}\right\} p = -2$ and $q = -3$ |
| **2.** | Plug the values for $p$ and $q$ into:<br>$(x+p)(x+q)$ | $(x-2)(x-3) = x^2 - 5x + 6$ |
| **3.** | Equate each part to 0<br>$(x+p) = 0$, $(x+q) = 0$,<br>and solve for $x$ | $\left.\begin{aligned} (x-2) &= 0 \\ (x-3) &= 0 \end{aligned}\right\} x = 2$ or $x = 3$ |

## Solving quadratic equations using the quadratic formula.

Solve: $3x^2-8x+4=0$

| | | |
|---|---|---|
| **1.** | Calculate the discriminant $\Delta$<br>$\Delta=b^2-4ac$ | $\Delta=(-8)^2-4\cdot3\cdot4=16$ |
| **2.** | How many solutions?<br>$\Delta>0\Rightarrow$ 2 solutions<br>$\Delta=0\Rightarrow$ 1 solution<br>$\Delta<0\Rightarrow$ no real solutions | $\Delta>0$, so 2 solutions |
| **3.** | Calculate $x$, use<br>$x=\dfrac{-b\pm\sqrt{\Delta}}{2a}$ | $x=\dfrac{8\pm\sqrt{16}}{2\cdot3}=\dfrac{8\pm4}{6}$<br>$\left.\begin{aligned}&=\dfrac{8-4}{6}=\dfrac{4}{6}\\&=\dfrac{8+4}{6}=2\end{aligned}\right\}\Rightarrow\quad x=\dfrac{2}{3}$ or $x=2$ |

By completing the square you can find the value of the *vertex* (the minimum or maximum). For the exam you will always be asked explicitly.

## Find the vertex by completing the square

$4x^2-2x-5=0$

| | | |
|---|---|---|
| **1.** | Move $c$ to the other side | $4x^2-2x=5$ |
| **2.** | Divide by $a$ | $x^2-\frac{1}{2}x=\frac{5}{4}$ |
| **3.** | Calculate $\left(\dfrac{x\text{ coeficient}}{2}\right)^2$ | $\left(\dfrac{-\frac{1}{2}}{2}\right)^2=\dfrac{1}{16}$ |
| **4.** | Add this term to both sides | $x^2-\frac{1}{2}x+\frac{1}{16}=\frac{5}{4}+\frac{1}{16}$ |
| **5.** | Factor perfect square, bring constant back | $\left(x-\frac{1}{4}\right)^2-\frac{21}{16}=0$<br>$\Rightarrow$ minimum point$=\left(\frac{1}{4},-\frac{21}{16}\right)$ |

Other forms: $y=a(x-h)^2+k$ vertex $(h,k)$ and $y=a(x-p)(x-q)$, $x$ intercepts: $(p,0)(q,0)$.

## 2.1.3 Functions with asymptotes

**Asymptote** a straight line that a curve approaches, but never touches.

A single function can have multiple asymptotes: horizontal, vertical and in rare cases diagonal. Functions that contain the variable $(x)$ in the denominator of a fraction will always have asymptotes, as well as exponential and logarithmic functions.

### Vertical asymptotes

Vertical asymptotes occur when the denominator is zero, as dividing by zero is undefinable. Therefore if the denominator contains $x$ and there is a value for $x$ for which the denominator will be 0, we get a vertical asymptote.

**Example.**

In the function $f(x) = \frac{x}{x-4}$, when $x = 4$, the denominator is 0 so there is a vertical asymptote.

### Horizontal asymptotes

Horizontal asymptotes are the value that a function tends to as $x$ become really big or really small; technically: to the limit of infinity, $x \to \infty$. When $x$ is large other parts of the function not involving $x$ become insignificant and so can be ignored.

**Example.**

In the function $f(x) = \frac{x}{x-4}$, when $x$ is small the 4 is important.

$x = 10$ $10 - 4 = 6$

But as $x$ gets bigger the 4 becomes increasingly insignificant

$x = 100$ $100 - 4 = 96$

$x = 10000$ $10000 - 4 = 9996$

Therefore as we approach the limits we can ignore the 4.

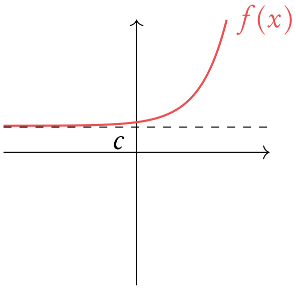
$$\lim_{x \to \infty} f(x) = \frac{x}{x} = 1$$

So there is a horizontal asymptote at $y = 1$.

### Exponential and logarithmic functions

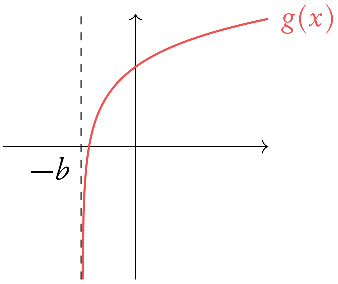
Exponential functions will always have a horizontal asymptote and logarithmic functions will always have a vertical asymptote, due to the nature of these functions. The position of the asymptote is determined by constants in the function.

Exponential



$f(x) = a^x + c$
where $a$ is a positive number
(often e)

Logarithmic



$g(x) = \log_a(x + b)$

## 2.1.4 Describing functions

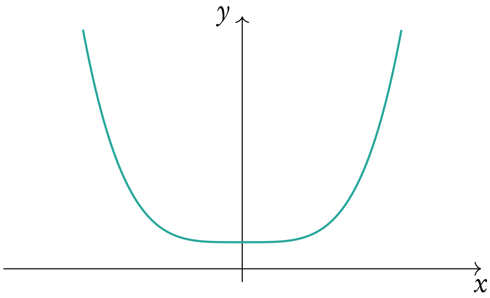
### Even and odd functions

When $f(x)=f(-x)$ we describe the function as *even* or a graph symmetrical over the $y$-axis.

Example.

**An even function: $f(x)=x^4+2$**

Testing algebraically substitute $(-x)$: $f(-x)=-x^4+2=x^4+2$



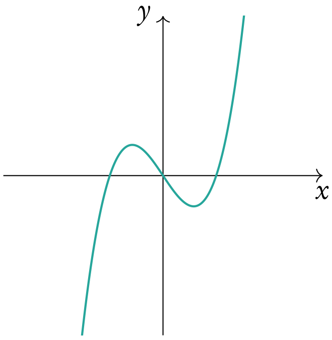
The graph is symmetrical over the $y$-axis.

When $-f(x)=f(-x)$ we describe the function as *odd* or a graph has rotational symmetry with respect to the origin.

Example.

**An odd function: $f(x)=x^3-x$**

Testing algebraically. $-f(x)=-x^3+x$.
Substitute $(-x)$: $f(-x)=-x^3+x=-x^3+x=-f(x)$

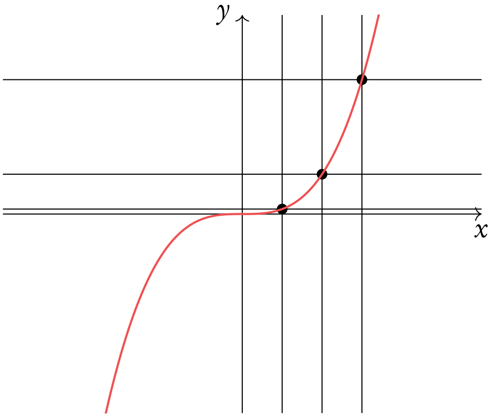


The graph has a rotational symmetry with respect to the origin.

## One to one function

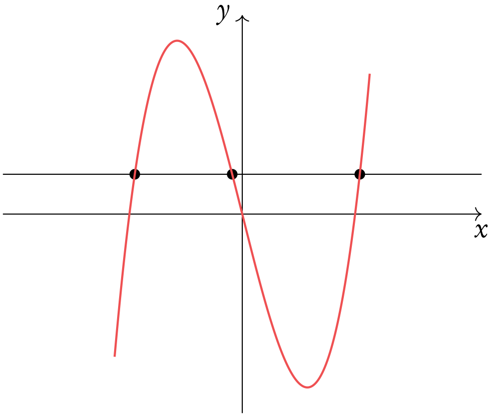
A one to one function is a function for which every element of the range of the function correspond to exactly one element of the domain.

Can be tested with horizontal and vertical line test.



## Many to one functions

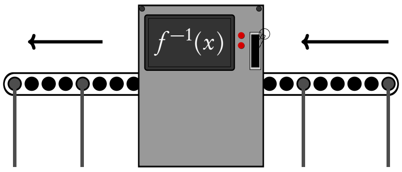
A meny-to-one function is a defined as a function where there are $y$-values that have more than one $x$-value mapped onto them.



## 2.2 Rearranging functions

### 2.2.1 Inverse functions, $f^{-1}(x)$

Inverse functions are the reverse of a function. Finding the input $x$ for the output $y$. You can think of it as going backwards through the number machine



This is the same as reflecting a graph in the $y = x$ axis.

**Finding the inverse function.**

$f(x) = 2x^3 + 3$, find $f^{-1}(x)$

| | | |
|---|---|---|
| **1.** | Replace $f(x)$ with $y$ | $y = 2x^3 + 3$ |
| **2.** | Solve for $x$ | $y - 3 = 2x^3$<br>$\Rightarrow \frac{y-3}{2} = x^3$<br>$\Rightarrow \sqrt[3]{\frac{y-3}{2}} = x$ |
| **3.** | Replace $x$ with $f^{-1}(x)$ and $y$ with $x$ | $\sqrt[3]{\frac{x-3}{2}} = f^{-1}(x)$ |

### 2.2.2 Composite functions

Composite functions are combination of two functions.

$$(f \circ g)(x) \qquad \text{means } f \text{ of } g \text{ of } x$$

To find the composite function above substitute the function of $g(x)$ into the $x$ of $f(x)$.

**Example.**

Let $f(x) = 2x + 3$ and $g(x) = x^2$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x)$: replace $x$ in the $f(x)$ function with the entire $g(x)$ function

$$(2g(x)) + 3 = 2x^2 + 3$$

$(g \circ f)(x)$: replace $x$ in the $g(x)$ function with the entire $f(x)$ function

$$(f(x))^2 = (2x + 3)^2$$

## 2.2.3 Transforming functions

By adding and/or multiplying by constants we can transform a function into another function.

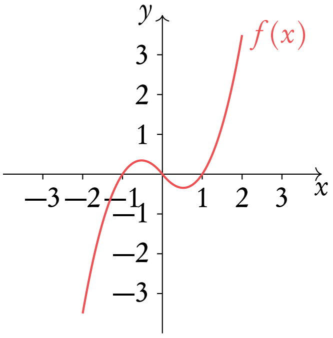
| Change to $f(x)$ | Effect |
|---|---|
| $f(x)+a$ | Move graph $a$ units upwards |
| $f(x+a)$ | Move graph $a$ units to the left |
| $a\cdot f(x)$ | Vertical stretch by factor $a$ |
| $f(a\cdot x)$ | Horizontal stretch by factor $1/a$ |
| $-f(x)$ | Reflection in $x$-axis |
| $f(-x)$ | Reflection in $y$-axis |

Exam hint: describe the transformation with words as well to guarantee marks.

### Transforming functions $f(x) \to af(x+b)$

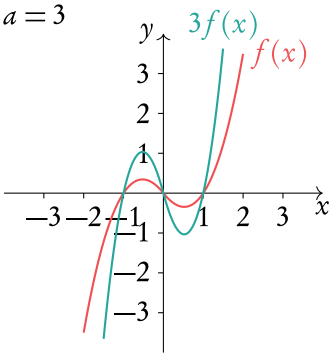
Given $f(x) = \frac{1}{4}x^3 + x^2 - \frac{5}{4}x$, draw $3f(x-1)$.

**1.** Sketch $f(x)$



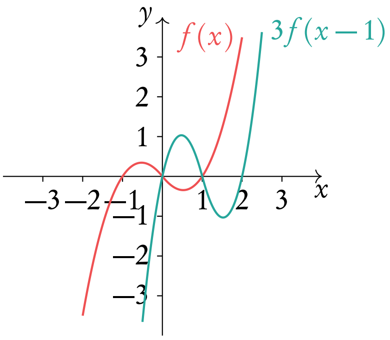
**2.** Stretch the graph by the factor of $a$

$a = 3$



**3.** Move graph by $-b$

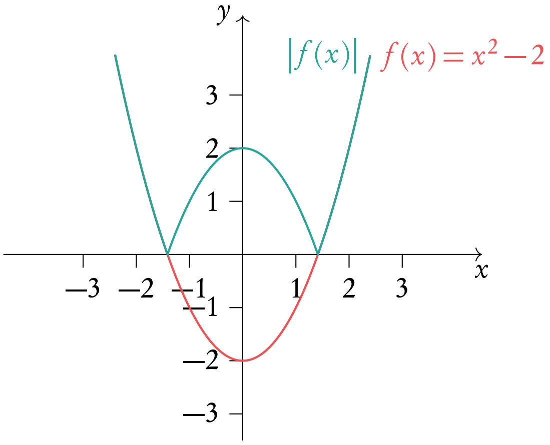
Move graph by 1 to the *right*



Example.

**Absolute value: $|f|$**

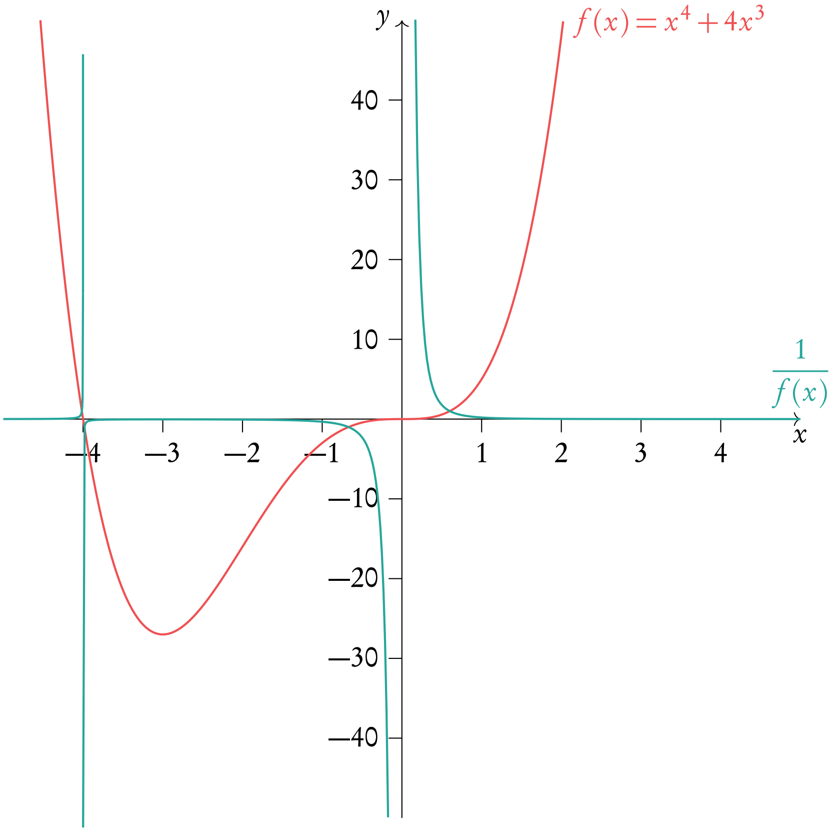
$f(x)=x^2-2 \Rightarrow |f(x)|=?.$



Example.

**Reciprocal: $\frac{1}{f(x)}$**

$f(x)=x^4+4x^3$ so: $\frac{1}{f(x)}=\frac{1}{x^4+4x^3}$

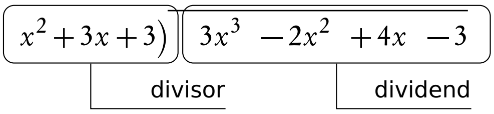


## 2.2.4 Polynomial long division

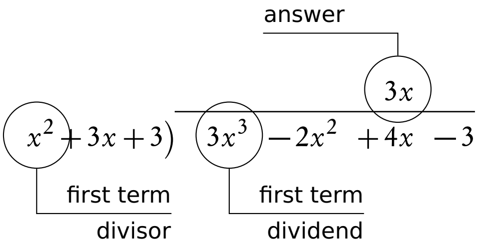
When we need to divide one polynomial by another we use *polynomial long division*. The number to be divided is called the ‘dividend’. The number which divides it is called ‘divisor’.

### Polynomial long division

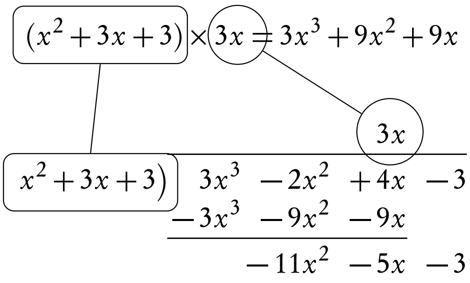
Divide $3x^3 - 2x^2 + 4x - 3$ by $x^2 + 3x + 3$:



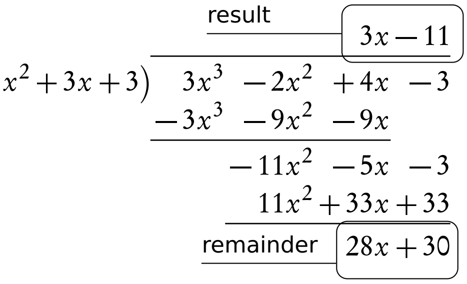
**1.** Divide the first term of the dividend by the first term of the divisor



**2.** Multiply the divisor by this answer and subtract this from our dividend



**3.** Divide the result of the substituition by the first term of the divisor. Repeat the process until this is no longer possibile



**4.** Write the answer:

result $+ \dfrac{\text{remainder}}{\text{divisor}}$

$$3x - 11 + \frac{28x+30}{x^2+3x+3}$$

## 2.3 The factor and remainder theorem

**Remainder theorem** when we divide a polynomial $f(x)$ by $x - c$ the remainder $r$ equals $f(c)$

Let's say

$$f(x) \div (x - c) = q(x) + r$$

where $r$ is the remainder. We also know

$$f(x) = (x - c)q(x) + r$$

If we now substitute $x$ with $c$

$$f(c) = (c - c)q(c) + r$$

but $c - c = 0$, therefore

$$f(c) = r$$

**Factor theorem** when $f(c) = 0$ then $x - c$ is a factor of the polynomial

# VECTORS

# 3

## Table of contents & cheatsheet

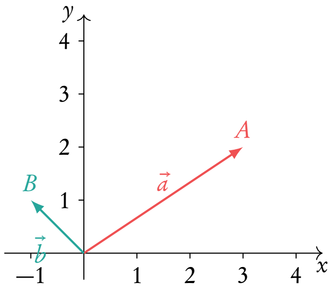
### Definitions

| | |
|---|---|
| Vector | a geometric object with *magnitude* (length) and *direction*, represented by an *arrow*. |
| Collinear points | points that lie on the same line |
| Unit vector | vector with magnitude 1 |
| Base vector | $\vec{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$ |

### 3.1. Working with vectors 34

**Vector from point $O$ to point $A$:** $\vec{OA}=\vec{a}=\begin{pmatrix}3\\2\end{pmatrix}$

**Vector from point $O$ to point $B$:** $\vec{OB}=\vec{b}=\begin{pmatrix}-1\\1\end{pmatrix}$



Can be written in two ways:

$$\vec{a}=\begin{pmatrix}3\\2\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

$$\vec{a}=3i+2j+0k=3i+2j$$

**Length of $\vec{a}$:** $|\vec{a}|=\sqrt{x^2+y^2}=\sqrt{3^2+2^2}=\sqrt{13}$

**Addition & multiplication:** $\vec{a}+2\vec{b}=\begin{pmatrix}3\\2\end{pmatrix}+2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}-2\\2\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}$

**Subtraction:** $\vec{a}-\vec{b}=\begin{pmatrix}3\\2\end{pmatrix}-\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$

### 3.2. Equations of lines 36

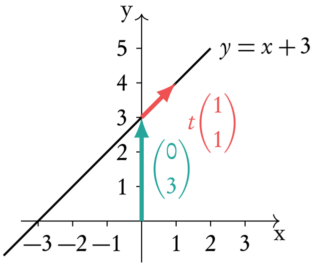
Example of a line:

$$r=\begin{pmatrix}0\\3\end{pmatrix}+t\begin{pmatrix}1\\1\end{pmatrix}$$

direction vector

parameter

position vector



### 3.3. Dot product 38

The dot product of two vectors $\vec{c}\cdot\vec{d}$ can be used to find the angle between them.

Let $\vec{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}, \vec{d}=\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$:

$$\vec{c}\cdot\vec{d}=|\vec{c}||\vec{d}|\cos\theta$$

$$\vec{c}\cdot\vec{d}=c_1d_1+c_2d_2+c_3d_3$$

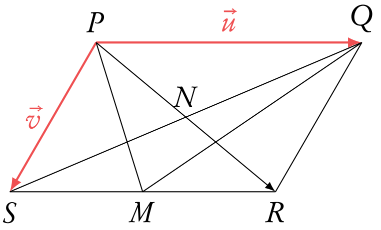
## 3.1 Working with vectors

Vectors are a geometric object with a *magnitude* (length) and *direction*. They are represented by an *arrow*, where the arrow shows the direction and the length represents the magnitude.

So looking at the diagram we can see that vector $\vec{u}$ has a greater magnitude than $\vec{v}$. Vectors can also be described in terms of the points they pass between. So

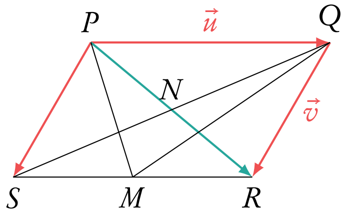
$$\begin{cases} \vec{u} = \vec{PQ} \\ \vec{v} = \vec{PS} \end{cases}$$

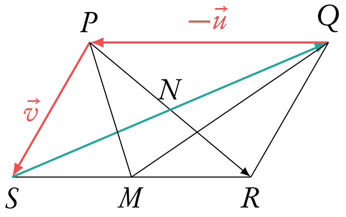
with the arrow over the top showing the direction.

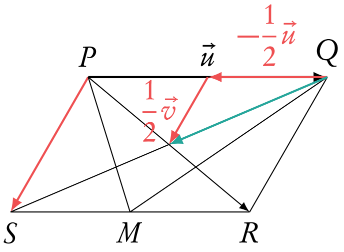


You can use vectors as a geometric algebra, expressing other vectors in terms of $\vec{u}$ and $\vec{v}$. For example

$$\vec{PR} = \vec{u} + \vec{v} \qquad \vec{QS} = -\vec{u} + \vec{v} \qquad \vec{QN} = \frac{1}{2}(-\vec{u} + \vec{v})$$







This may seem slightly counter-intuitive at first. But if we add in some possible figures you can see how it works. If $\vec{u}$ moves 5 units to the left and $\vec{v}$ moves 1 unit to the right (−left) and 3 units down.

Then $\vec{PR} = \vec{u} + \vec{v} =$ 5 units to the left −1 unit to the right and 3 units down = 4 units to the left and 3 units down.

## 3.1.1 Vectors with value

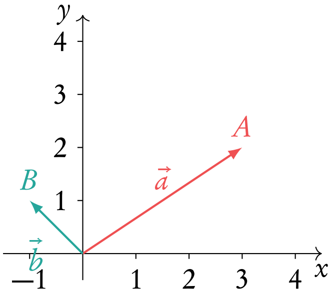
Formally the value of a vector is defined by its direction and magnitude within a 2D or 3D space. You can think of this as the steps it has to take to go from its starting point to its end, moving only in the $x$, $y$ and $z$ axis.

**Vector from point $O$ to point $A$:**

$$\vec{OA} = \vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

**Vector from point $O$ to point $B$:**

$$\vec{OB} = \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Vectors can be written in two ways:

Note: unless told otherwise, answer questions in the form used in the question.

1. $\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where the top value is movement in the $x$-axis. Then the next is movement in the $y$ and finally in the $z$. Here the vector is in 2D space as there is no value for the $z$-axis.
2. as the sum of the three base vectors:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here $\vec{i}$ is moving 1 unit in the $x$-axis, $\vec{j}$ 1 unit in the $y$-axis and $\vec{k}$ 1 unit in the $z$-axis.

$$\vec{a} = 3i + 2j + 0k = 3i + 2j$$

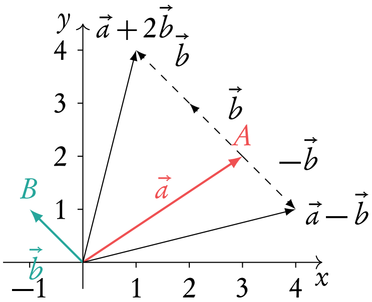
When we work with vectors we carry out the mathematical operation in each axis separately. So $x$-values with $x$-values and so on.

**Addition & multiplication:**

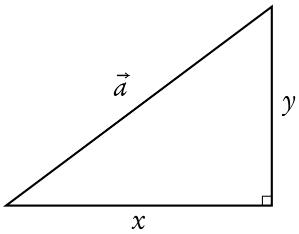
$$\vec{a} + 2\vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

**Subtraction:**

$$\vec{a} - \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$



However it must be remembered that vector notation does not give us the actual length (magnitude) of the vector. To find this we use something familiar.



**Length of $\vec{a}$:**

$$\begin{aligned}|\vec{a}| &= \sqrt{x^2+y^2}\\ &= \sqrt{3^2+2^2} = \sqrt{13}\end{aligned}$$

Sometimes you will be asked to work with unit vectors. These are vectors with a magnitude of 1. We can convert all vectors to unit vectors.

### Determine the unit vector $\hat{a}$ in the direction of any vector $\vec{a}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix}$$

## 3.2 Equations of lines

We can further divide vectors into two types:

position vectors vectors from the origin to a point,

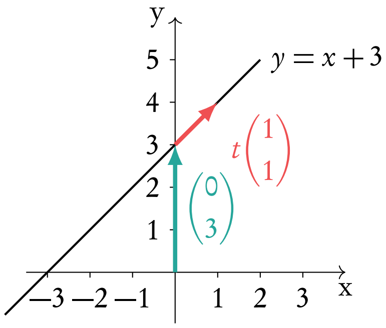
e.g. $P = (-1,3) \Rightarrow \vec{P} = \begin{pmatrix}-1\\3\end{pmatrix}$.

direction vectors vectors that define a direction.

Using both we can define lines in terms of vectors.
Example of a line:

$$r = \begin{pmatrix}0\\3\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}$$

direction vector
parameter
position vector



Note the position vector can go to any where on the line. So in this example we could also use $(-3,0)$ or $(1,4)$. Equally the direction vector can be scaled. So we could used $(2,2)$, $(30,30)$, ...

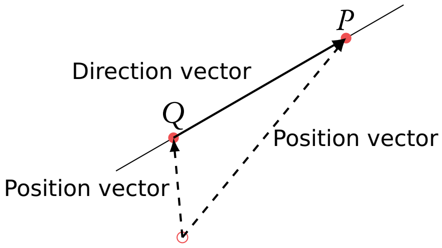
Because of this parallel lines will have direction vectors with the same ratio but not necessarily in exact numbers.

**Parallel lines:** direction vector of $L_1$ = direction vector of $L_2$ × constant

Questions often deal with points and or multiple lines. It is worth making a sketch to help understand the question.

## Finding a line passing through two points.

Find the equation of the line passing through points $P=(1,3,2)$ and $Q=(0,-1,4)$.
Does point $R=(-2,9,1)$ lie on the line?



Note this can go either way from $Q$ to $P$ of $P$ to $Q$.

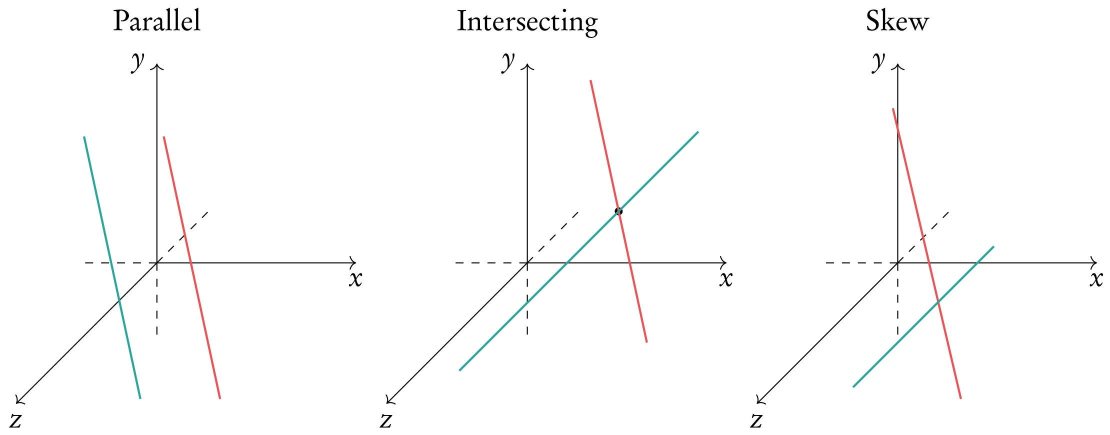
| | | |
|---|---|---|
| **1.** | Write points as position vectors | $\vec{P}=\begin{pmatrix}1\\3\\2\end{pmatrix}, \vec{Q}=\begin{pmatrix}0\\-1\\4\end{pmatrix}$ |
| **2.** | Direction vector<br>= vector between points | $\begin{pmatrix}0-1\\-1-3\\4-2\end{pmatrix}=\begin{pmatrix}-1\\-4\\2\end{pmatrix}$ |
| **3.** | Choose $\vec{P}$ or $\vec{Q}$ as position vector | $r=\begin{pmatrix}1\\3\\2\end{pmatrix}+t\begin{pmatrix}-1\\-4\\2\end{pmatrix}$ |
| **4.** | Equate $\vec{R}$ and the line $r$.<br>If there is no contradiction,<br>$R$ lies on $r$ | $\begin{pmatrix}-2\\9\\1\end{pmatrix}=\begin{pmatrix}1\\3\\2\end{pmatrix}+t\begin{pmatrix}-1\\-4\\2\end{pmatrix}$<br>$\Rightarrow -2=1-t \Rightarrow t=3$<br>$\Rightarrow 9=3-4t \Rightarrow 9\neq 3-12$<br>$\Rightarrow R$ does not lie on the line. |

## Finding the intersection of two lines.

Find the intersection for $r_1=\begin{pmatrix}2\\1\\0\end{pmatrix}+s\begin{pmatrix}-3\\1\\4\end{pmatrix}$ and $r_2=\begin{pmatrix}-1\\3\\7\end{pmatrix}+t\begin{pmatrix}3\\0\\-1\end{pmatrix}$

| | | |
|---|---|---|
| **1.** | Equate<br>write simultaneous equations | $\begin{cases}2-3s=-1+3t\\1+s=3\end{cases}$ |
| **2.** | Solve | $s=2,\ t=-1$ |
| **3.** | Substitute back into $r_1$ or $r_2$ | $\begin{pmatrix}2-3(2)\\1+2\\4(2)\end{pmatrix}=\begin{pmatrix}-4\\3\\8\end{pmatrix}$ |

If one considers two lines in a three-dimensional graph, then there are three ways in which they can interact:


Parallel

Intersecting

Skew

If direction vectors defining a line aren't multiples of one another, then the lines can either be intersectiong or skew. One can find out if the lines intersect by equationg the vector equations and attempting to solve the set of equations (remember: one needs as many equations as variable to solve).

If one can't find a point of intersection, then the lines are skew.

## 3.3 Dot (scalar) product

DB 4.2

Learn to add the following statement to questions asking "*are they perpendicular?*". $\vec{c}\cdot\vec{d} = 0$ therefore $\cos x = 0$, therefore $x = 90^\circ$. Lines are perpendicular. Of course, when lines are not perpendicular replace all $=$ with $\neq$.

The dot product of two vectors $\vec{c}\cdot\vec{d}$ can be used to find the angle between them. Let

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\vec{c}\cdot\vec{d} = |\vec{c}||\vec{d}|\cos\theta$$
$$\vec{c}\cdot\vec{d} = c_1d_1 + c_2d_2 + c_3d_3$$

**Finding the angle between two lines.**
**(Often are these two vectors perpendicular)**

Find the angle between $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$.

| | | |
|---|---|---|
| **1.** | Find $\vec{c} \cdot \vec{d}$ in terms of components | $\vec{c} \cdot \vec{d} = 2 \times 8 + 3 \times 1 + (-1) \times 3 = 16$ |
| **2.** | Find $\vec{c} \cdot \vec{d}$ in terms of magnitudes | $\vec{c} \cdot \vec{d} = \sqrt{2^2 + 3^2 + (-1)^2} \times \sqrt{8^2 + 1^2 + 3^2} \times \cos\theta = \sqrt{14}\sqrt{74}\cos\theta$ |
| **3.** | Equate and solve for $\theta$ | $16 = \sqrt{14}\sqrt{74}\cos\theta$<br>$\Rightarrow \cos\theta = \dfrac{16}{\sqrt{14}\sqrt{74}}$<br>$\Rightarrow \theta = 60.2°$ |

Note: when $\theta = 90°$ (perpendicular vectors), $\cos(90°) = 0 \Rightarrow \vec{c} \cdot \vec{d} = 0$

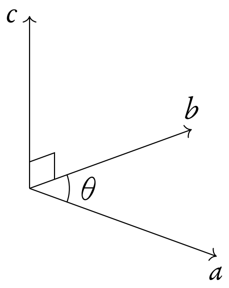
## 3.4 Cross (vector) product

The cross product of two vectors produces a third vector which is perpendicular to both of the two vectors. As the result is a vector, it is also called the *vector product*.

There are two methods to find the cross product:

1. $a \times b = |a||b|\sin\theta \mathrm{n}$ where $\theta$ is the angle between $a$ and $b$ and n is a unit vector in the direction of $c$.
2. $x = a \times b$, where

$$c_1 = a_2 b_3 - a_3 b_2$$
$$c_2 = a_3 b_1 - a_1 b_3$$
$$c_3 = a_1 b_2 - a_2 b_1$$



**Example.**

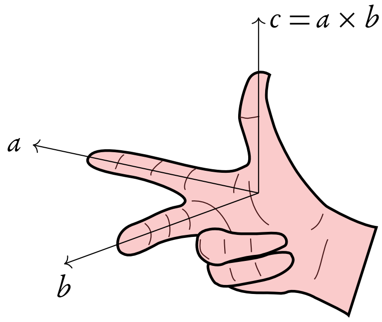
Find the cross product of $a \times b$.
$a = (2,3,4)$, $b = (5,6,7)$.

$$c_1 = 3 \times 7 - 4 \times 6 = -3$$
$$c_2 = 4 \times 5 - 2 \times 7 = 6$$
$$c_3 = 2 \times 6 - 3 \times 5 = -3$$

$\Rightarrow a \times b = (-3,6,-3)$

Remember the cross product is not commutative, so $a \times b \neq b \times a$.

You can check the direction of $c$ with the right hand rule:



## 3.5 Equation of a plane

Planes are 2 dimensional surface in 3 dimensional space. They can be defined by a position vector and 2 direction vectors (which are not parallel)

$$r = \underbrace{a}_{\text{position vector}} + \overbrace{\lambda \underbrace{\begin{pmatrix}x\\y\\z\end{pmatrix} + \mu \begin{pmatrix}x\\y\\z\end{pmatrix}}_{\text{direction vectors}}}^{\text{parameter}}$$

or in a cartesian form

$$ax + by + cz = d$$

where $d$ is a constant.

**Find a cartesian equation of a plane from 3 points.**

Find a cartesian equation of the plane $P$ containing $A(2,0,-3)$, $B(1,-1,6)$ and $C(5,5,0)$.

| | | |
|---|---|---|
| **1.** | Find two lines | $AB = B - A = -i - j + ak$<br>$AC = C - A = 3i + 5j + 3k$ |
| **2.** | Take the cross product of these two lines | $AB \times AC = -48i + 30j - 2k$ |
| **3.** | Substitute a point back in to the cross product (here $A$) | $-48(x-2) + 30(y) - 2(z+3) = 0$<br>$-48x + 30y - 2z = -90$<br>$24x - 15y + z = 45$ |

## 3.5.1 Line and plane

Lines can instersect with a plane in 3 ways:

| | |
|---|---|
| 1. Parallel to the plane | 0 solutions |
| 2. Intersect the plane | 1 solution |
| 3. Lie on the plane | infinite solutions |

### Does a line intersect a plane?

The line $L_1$ passes through the points $(1, 0, 1)$ and $(4, -2, 2)$. Does it intersect the plane $x + y + 2 = 6$.

| | | |
|---|---|---|
| **1.** | Find parameter rapresentation of the line | $x = 1 + 3\lambda$<br>$y = -2\lambda$<br>$z = 1 + \lambda$ |
| **2.** | Put into the equation for the plane | $(1 + 3\lambda) + (-2\lambda) + (1 + \lambda) = 6$ |
| **3.** | Solve for $\lambda$ | $2 + 2\lambda = 6$<br>$2\lambda = 4$<br>$\lambda = 2$ |
| **4.** | Find point of intersection | $x = 1 + 3(2) = 7$<br>$y = -2(2) = -4$<br>$z = 1 + (2) = 3$<br>$\begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} =$ point of intersection |

## 3.5.2 Plane and plane

### Intersection of two planes

When two planes intersect, they will intersect along a line.

#### Finding line of intersection of two planes

Find the intersection of $x+y+z+1=0$ and $x+2y+3z+4=0$

| | | |
|---|---|---|
| **1.** | Check the equations for planes are in a cartesian form; move the constant to the other side | $\begin{cases} x+y+z=-1 & (1) \\ x+2y+3z=-4 & (2) \end{cases}$ |
| **2.** | Solve the system of equations to remove a variable | (1) − (2)<br>$\begin{array}{r} \left\{ \begin{array}{r} x+y+z=-1 \\ -x-2y-3z=-4 \end{array} \right. \\ \hline -y-2z=3 \end{array}$<br>$y=-3-2z$ |
| **3.** | Let $z=t$ | $\Rightarrow \quad y=-3-2t$. Rearrange:<br>$x=-1-y-z$<br>$x=-1-(-3-2t)-t$<br>$x=t+2$ |
| **4.** | Find the result | Intersection occurs at line<br>$(x,y,z)=(t+2,-2t-3,t)$ or<br>$r=\begin{pmatrix}2\\-3\\0\end{pmatrix}+\begin{pmatrix}1\\-2\\1\end{pmatrix}t$ |

### Intersection of three planes

Unless two or more planes are parallel, three planes will intersect at a point. If two are parallel there will be two lines of intersect. If all three are parallel, there will be no solutions.

We have three variables and three equations and se we can solve the system.

**Finding point of intersect of three planes**

Find the instersect of the three planes

$$\begin{aligned} x-3y+3z&=-4 && \text{(a)}\\ 2x+3y-z&=15 && \text{(b)}\\ 4x-3y-z&=19 && \text{(c)}\end{aligned}$$

**1.** Eliminate one variable in two pair of lines (here $z$)

$$\begin{aligned} \text{(b)}-\text{(c)}\Rightarrow&\quad -2x+6y=-4 && \text{(d)}\\ \text{(a)}+3\text{(b)}\Rightarrow&\quad 7x+6y=41 && \text{(e)}\end{aligned}$$

**2.** Eliminate antoher variable from these new lines (here $y$)

$$\begin{aligned} \text{(e)}-\text{(d)}\Rightarrow&\quad 9x=45\\ &\quad x=5\end{aligned}$$

**3.** Place the value into the equations to find values for $x$, $y$, and $z$

$$\begin{aligned} & & x&=5\\ \text{(d)}\quad & -2(5)+6y=-4 & \Rightarrow y&=1\\ \text{(a)}\quad & (5)-3(1)+3z=-4 & \Rightarrow z&=-2\end{aligned}$$

Point of intersection $(5,1,-2)$

## 3.5.3 Normal vector

By taking the cross product of the two direction vectors that define a plane, we can find the normal vector. This vector is perpendicular to the plane. In turn the normal vector can be used to show a line is parallel to the plane by using the dot product. If parallel $\mathrm{n}\cdot d=0$, where n is the normal vector and $d$ is the direction vector of a line.

# TRIGONOMETRY AND CIRCULAR FUNCTIONS

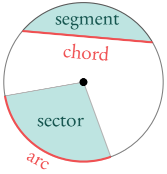
4

### Table of contents & cheatsheet

## 4.1. Basic trigonometry 46

radians $= \frac{\pi}{180°} \times$ degrees  degrees $= \frac{180°}{\pi} \times$ radians

Before each question make sure calculator is in correct setting: degrees or radians?

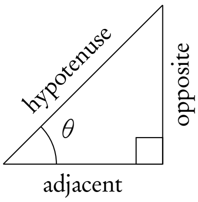


**Area of a sector** $= \frac{1}{2} r^2 \cdot \theta$

**Arc length** $= r \cdot \theta$

$\theta$ in radians, $r =$ radius.

**Right-angle triangle (triangle with 90° angle)**

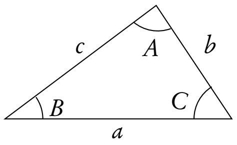


$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ **SOH**

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ **CAH**

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ **TOA**

**Non-right angle triangles**



**Sine rule:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

*Use this rule when you know:* 2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides).

**Cosine rule:** $c^2 = a^2 + b^2 - 2\,ab\cos C$

*Use this rule when you know:* 3 sides or 2 sides and the angle between them.

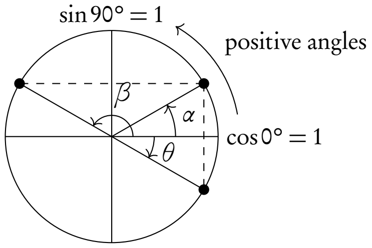
**Area of a triangle:** Area $= \frac{1}{2} ab \sin C$

*Use this rule when you know:* 3 sides or 2 sides and the angle between them.

**Three-figure bearings**

Direction given as an angle of a full circle. North is 000 and the angle is expressed in the clockwise direction from North. So East is 090, South is 180 and West 270.

## 4.2. Circular functions 51



| deg | 0° | 30° | 45° | 60° | 90° | 120° | 135° | 150° | 180° |
|---|---|---|---|---|---|---|---|---|---|
| **rad** | 0 | $\frac{1}{6}\pi$ | $\frac{1}{4}\pi$ | $\frac{1}{3}\pi$ | $\frac{1}{2}\pi$ | $\frac{2}{3}\pi$ | $\frac{3}{4}\pi$ | $\frac{5}{6}\pi$ | $\pi$ |
| **$\sin\theta$** | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| **$\cos\theta$** | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| **$\tan\theta$** | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | 0 |

### Trigonometric function $y = a\sin(bx + c) + d$

Amplitude: $a$

Period: $\frac{360°}{b}$ or $\frac{2\pi}{b}$

Horizontal shift: $c$

Vertical shift: $d$

### Trigonometric identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$2\sin\theta\cos\theta = \sin 2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

# 4.1 Basic trigonometry

This section offers an overview of some basic trigonometry rules and values that will recur often. It is worthwhile to know these by heart; but it is much better to understand how to obtain these values. Like converting between Celsius and Fahrenheit; you can remember some values that correspond to each other but if you understand how to obtain them, you will be able to convert any temperature.

## 4.1.1 Converting between radians and degrees

$$\text{radians} = \frac{\pi}{180°} \times \text{degrees}$$

$$\text{degrees} = \frac{180°}{\pi} \times \text{radians}$$

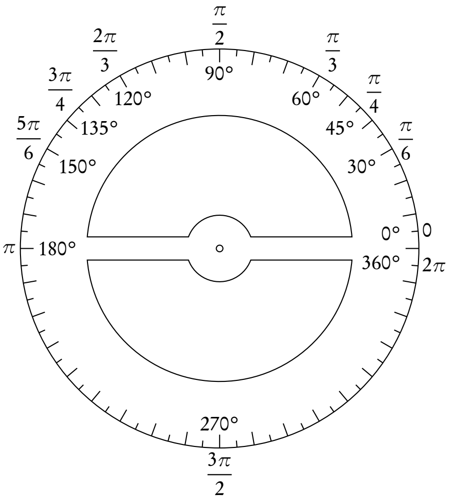


Table 4.1: Common radians/degrees conversions

| **Degrees** | 0° | 30° | 45° | 60° | 90° | 120° | 135° | 180° | 270° | 360° |
|---|---|---|---|---|---|---|---|---|---|---|
| **Radians** | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |

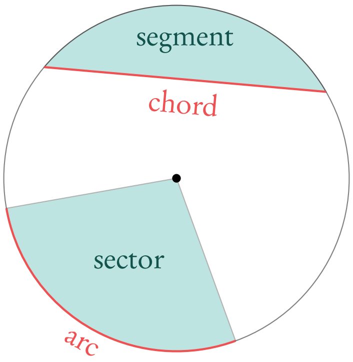
## 4.1.2 Circle formulas

DB 3.1

**Area of a sector** $= \frac{1}{2}r^2 \cdot \theta$

**Arc length** $= r \cdot \theta$

$\theta$ in radians, $r =$ radius.



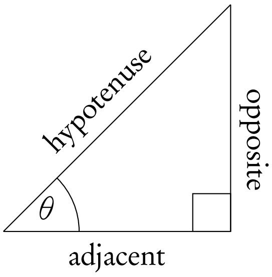
## 4.1.3 Right-angle triangles

$$a^2 = b^2 + c^2 \qquad \text{Pythagoras}$$

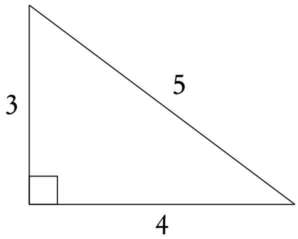
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{SOH}$$

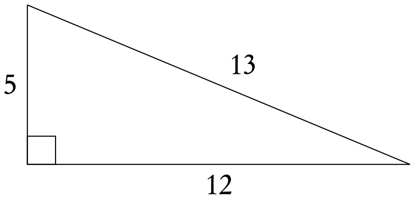
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{CAH}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{TOA}$$

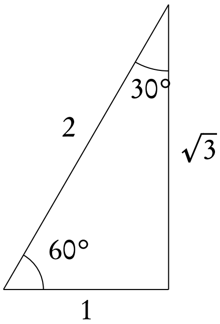


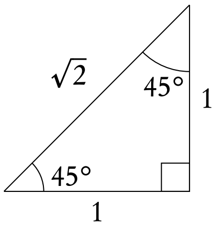
Two important triangles to memorize:





The IB loves asking questions about these special triangles which have whole numbers for all the sides of the right triangles.



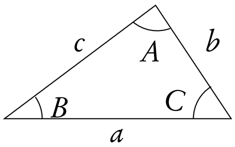


Note: these triangles can help you in finding the sin, cos and tan of the angles that you should memorize, shown in table 4.2 at page 52. Use SOH, CAH, TOA to find the values.

Read the question, does it specify if you are looking for an acute (less than 90°) or obtuse (more than 90°) angle. If not there may be 2 solutions. Exam hint: Use sketches when working with worded questions!

DB 3.6

## 4.1.4 Non-right angle triangles



To find any missing angles or side lengths in non-right angle triangles, use the *cosine* and *sine* rule. Remember that the angles in the triangle add up to 180°!

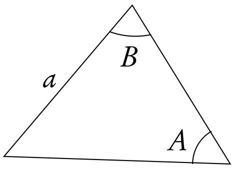
**Sine rule:** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

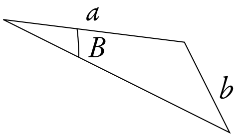
*Use this rule when you know:*

2 angles and a side
(not between the angles)

or

2 sides and an angle
(not between the sides)





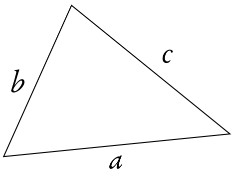
**Cosine rule:** $c^2 = a^2 + b^2 - 2\,a\,b\cos C$

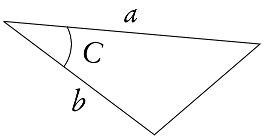
*Use this rule when you know:*

3 sides

or

2 sides and the angle between them





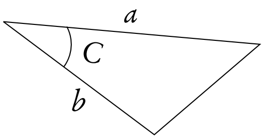
**Area of a triangle:** Area $= \dfrac{1}{2}a\,b\sin C$

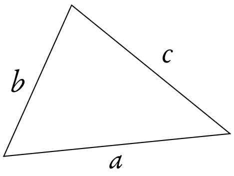
*Use this rule when you know:*

2 sides and the angle between them

or

3 sides
first you need to use cosine rule
to find an angle





Example.

$\triangle ABC$ : $A = 40°$, $B = 73°$, $a = 27\,\text{cm}$.

**Find $\angle C$.**

$$\angle C = 180° - 40° - 73° = 67°$$

**Find $b$.**

$$\begin{aligned} \frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{27}{\sin 40°} &= \frac{b}{\sin 73°} \\ b &= \frac{27}{\sin 40°} \cdot \sin 73° = 40.169 \approx 40.2\,\text{cm} \end{aligned}$$

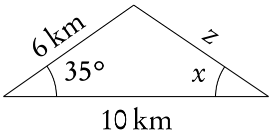
**Find $c$.**

$$\begin{aligned} \frac{c}{\sin C} &= \frac{a}{\sin A} \\ c &= \frac{27}{\sin 40°} \times \sin 67° = 38.7\,\text{cm} \end{aligned}$$

**Find the area.**

$$\begin{aligned} \text{Area} &= \frac{1}{2} \cdot 27 \cdot 40 \cdot 2 \cdot \sin 67° \\ &= 499.59 \approx 500\,\text{cm}^2 \end{aligned}$$

Example.



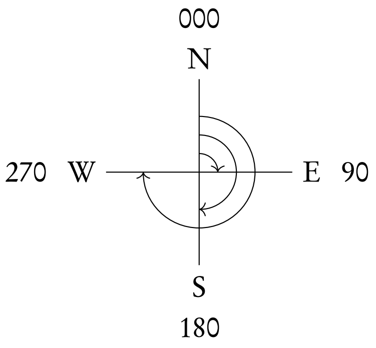
**Find $z$.**

$$\begin{aligned} z^2 &= 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 35° \\ z^2 &= 37.70 \\ z &= 6.14\,\text{km} \end{aligned}$$

**Find $\angle x$.**

$$\begin{aligned} \frac{6}{\sin x} &= \frac{6.14}{\sin 35°} \\ \sin x &= 0.56 \\ x &= \sin^{-1}(0.56) = 55.91° \end{aligned}$$

## 4.1.5 Three-figure bearings

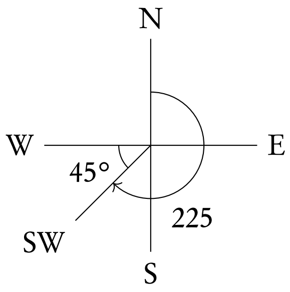


Three-figure bearings can be used to indicate compass directions on maps. They will be given as an angle of a full circle, so between 000 and 360. North is always marked as 000. Any direction from there can be expressed as the angle in the clockwise direction from North.

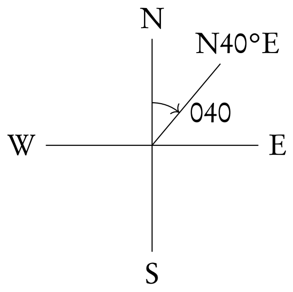
In questions on three-figure bearings, you are often confronted with quite a lot of text, so it is a good idea to first make a drawing. You may also need to create a right angle triangle and use your basic trigonometry.

**Example.**

SW: 45° between South and West = 225



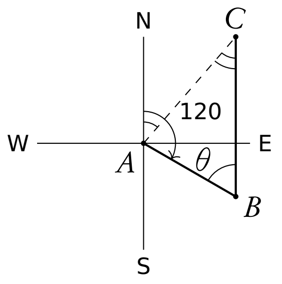
N40°E: 40° East of North = 040



**A ship left port $A$ and sailed $20\,\mathrm{km}$ in the direction 120.**

It then sailed north for $30\,\mathrm{km}$ to reach point $C$. How far from the port is the ship?

**1.** Draw a sketch



**2.** Find an internal angle of the triangle.

$\theta = 180° - 120° = 60° = C$

Similar angles between two parallel lines

**3.** Use cosine or sin rule.

(here cosine)

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos\theta$$

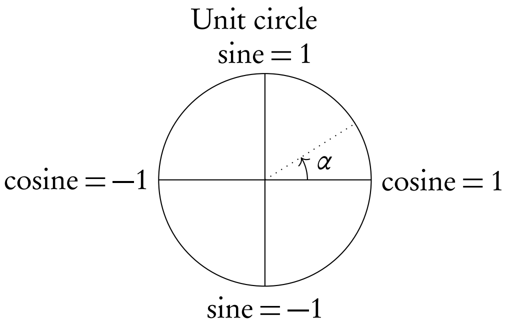
$$AC^2 = 20^2 + 30^2 - 2 \cdot 20 \cdot 30 \cdot \cos 60°$$

$$AC^2 = 400 + 900 - 2 \cdot 20 \cdot 30 \cdot \frac{1}{2}$$

$$AC = \sqrt{400 + 900 - 600} = \sqrt{700}$$

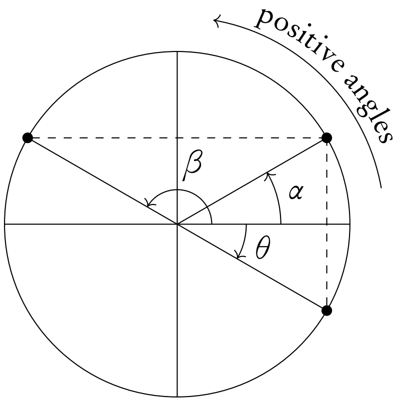
# 4.2 Circular functions

## 4.2.1 Unit circle



The unit circle is a circle with a radius of 1 drawn from the origin of a set of axes. The $y$-axis corresponds to *sine* and the $x$-axis to *cosine*; so at the coordinate $(0, 1)$ it can be said that cosine $= 0$ and sine $= 1$, just like in the $\sin x$ and $\cos x$ graphs when plotted.

The unit circle is particularly useful to find all the solutions to a trigonometric equation within a certain domain. As you can see from their graphs, functions with $\sin x$, $\cos x$ or $\tan x$ repeat themselves every given period; this is why they are also called *circular functions*. As a result, for each $y$-value there is an infinite amount of $x$-values that could give you this output. This is why questions will give you a set domain that limits the range of $x$-values you should consider in your calculations or represent on your sketch (e.g. $0° \leq x \leq 360°$).

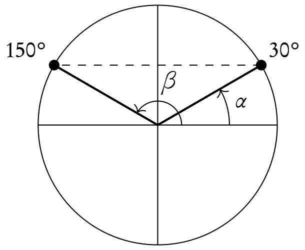


Relations between sin, cos and tan:

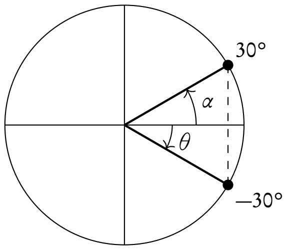
- $\alpha$ and $\beta$ have the same sine
- $\alpha$ and $\theta$ have the same cosine
- $\beta$ and $\theta$ have the same tangent

**Example.**

$\sin 30° = \sin 150°$



$\cos 30° = \cos 330°$



$\tan 150° = \tan 330°$

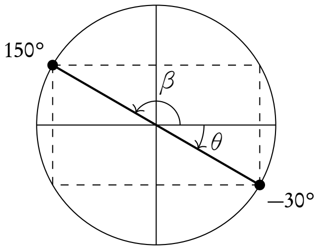
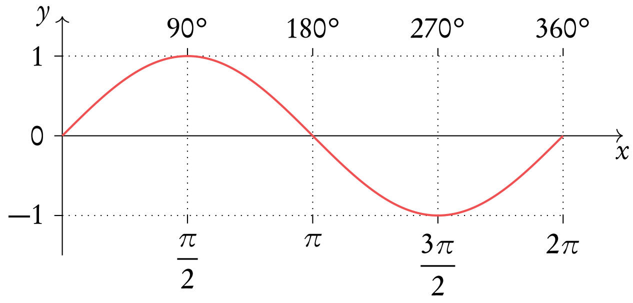


Table 4.2: Angles to memorize

| **deg** | 0° | 30° | 45° | 60° | 90° | 120° | 135° | 150° | 180° |
|---|---|---|---|---|---|---|---|---|---|
| **rad** | 0 | $\frac{1}{6}\pi$ | $\frac{1}{4}\pi$ | $\frac{1}{3}\pi$ | $\frac{1}{2}\pi$ | $\frac{2}{3}\pi$ | $\frac{3}{4}\pi$ | $\frac{5}{6}\pi$ | $\pi$ |
| **sin** $\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| **cos** $\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{\sqrt{2}}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| **tan** $\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | 0 |

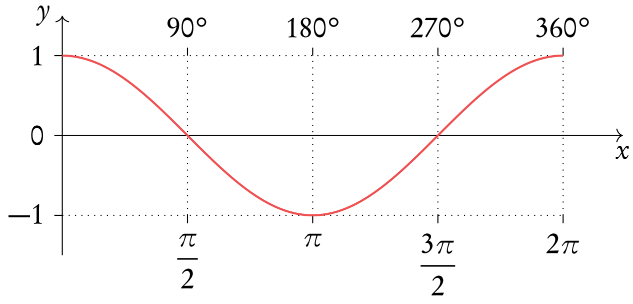
## 4.2.2 Graphs: trigonometric functions

### $\sin x$



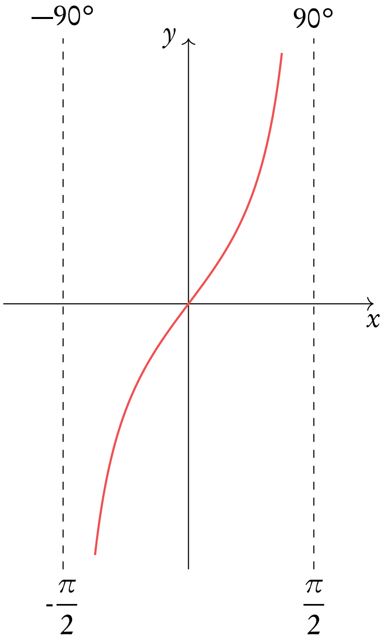
Domain: $x \in \mathbb{R}$
Amplitude: $-1 \leq y \leq 1$
Period: $2n\pi$, $n \cdot 360°$, with $n \in \mathbb{Z}$

### $\cos x$



Domain: $x \in \mathbb{R}$
Amplitude: $-1 \leq y \leq 1$
Period: $2n\pi$, $n \cdot 360°$, with $n \in \mathbb{Z}$

### $\tan x$



Domain: $x \in \mathbb{R}, x \neq \frac{\pi}{2} + k\pi,$ with $k \in \mathbb{Z}$
Amplitude: $-\infty < y < \infty$
Period: $n\pi$, $n \cdot 180°$, with $n \in \mathbb{Z}$

## 4.2.3 Transformations

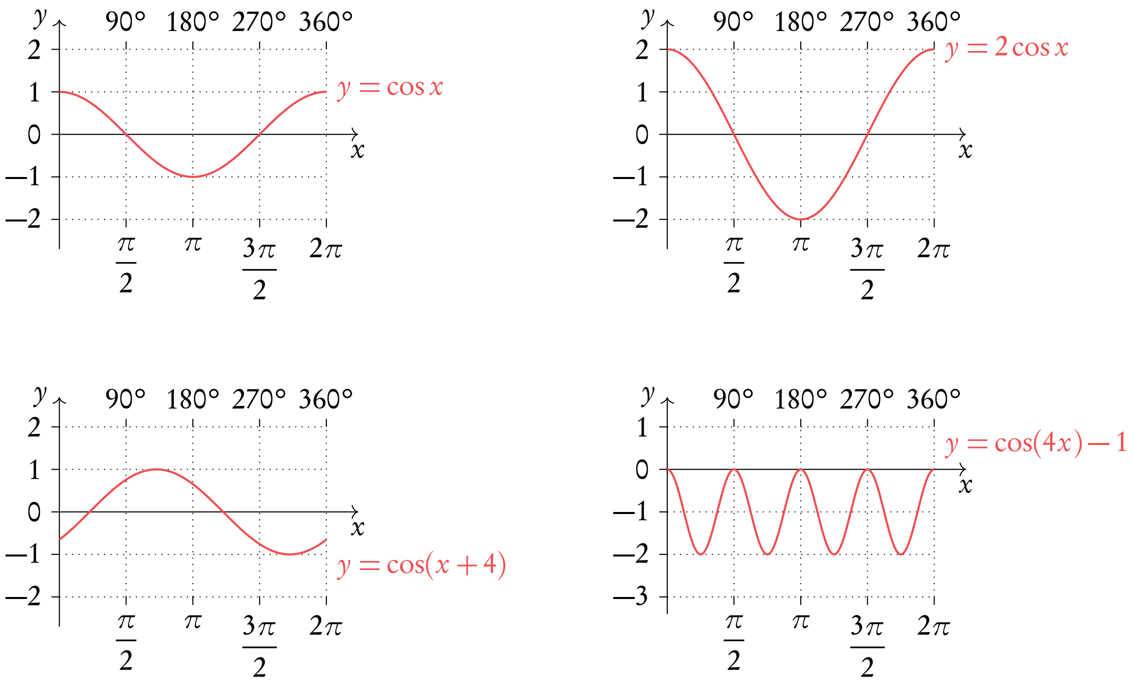
Besides the transformations in the functions chapter, trigonometric functions have some transformations with their own particular names. For a trigonometric function, the vertical stretch on a graph is determined by its amplitude, the horizontal stretch by its period and an upward/downward shift by its axis of oscillation.

A trigonometric function, given by $y = a\sin(bx + c) + d$, has:

- an amplitude $a$;
- a period of $\frac{360°}{b}$ or $\frac{2\pi}{b}$;
- a horizontal shift of $+c$ *to the left*, in degrees or radians;
- vertical shift of $+d$ *upwards*, oscillates around $d$.

**Example.**

**Transformations of $y = \cos x$.**



## 4.2.4 Identities and equations

In order to solve trigonometric equations, you will sometimes need to use identities. Identities allow you to rewrite your equation in a way that will make it easier to solve algebraically.

DB 3.2 & 3.3

### Trigonometric identity

$$\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ \sin^2\theta + \cos^2\theta &= 1 \\ 2\sin\theta\cos\theta &= \sin 2\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \end{aligned}$$

**Solving equations with trigonometric identities**

Solve $2\cos^2 x + \sin x = 1$, $0° \leq x \leq 360°$.

| | | |
|---|---|---|
| **1.** | Identify which identity from the databook to use. Note you are always aiming to get an equation with just, sin, cos or tan. | Here we could use either $\sin^2\theta + \cos^2\theta = 1$ or $\cos^2\theta - \sin^2\theta = \cos 2\theta$. We will use the first so that we get an equation with just sin. |
| **2.** | Rearrange identity and substitute into equation. | $\cos^2\theta = 1 - \sin^2\theta$<br>$2(1 - \sin^2 x) + \sin x = 1$<br>$2 - 2\sin^2 x + \sin x = 1$<br>$-2\sin^2 x + \sin x + 1 = 0$ |
| **3.** | Solve for $x$. Giving answers within the stated range. Recognise that here the eqauation looks like a quadratic equation. | Substitue $u$ for $\sin x$:<br>$-2u^2 + u + 1 = 0$<br>$(-2u - 1)(u - 1) = 0$<br>$u = \sin x \Rightarrow 1 \qquad x \Rightarrow 90°$<br>$u = \sin x \Rightarrow -0.5 \qquad x \Rightarrow 210°$ or $330°$ |

### Double angle and half angle formulae

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \pm \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \pm \tan A \tan B} \\
\cos(2a) &= \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \\
\sin(2a) &= 2\sin a \cos a \\
\tan(2a) &= \frac{2\tan a}{1 - \tan^2 a}
\end{aligned}$$

From the double angle we can obtain half angles.

$$\begin{aligned}
\cos a = \cos^2\left(\frac{a}{2}\right) - \sin^2\left(\frac{a}{2}\right) = 2\cos^2\left(\frac{a}{2}\right) - 1 = 1 - 2\sin^2\left(\frac{a}{2}\right) \\
\sin a = 2\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right) \\
\tan a = \frac{2\tan\left(\frac{a}{2}\right)}{1 - \tan^2\left(\frac{a}{2}\right)}
\end{aligned}$$

## 4.2.5 Inverse and reciprocal trigonometric functions

DB

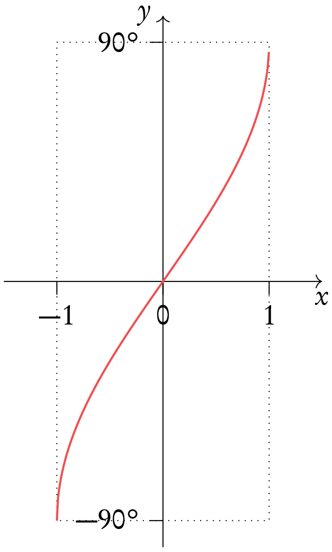
### Inverse trigonometric functions

The inverse of a trigonometric function is useful for finding an angle. You should already be familiar with carrying this operation out on a calculator.

$$\sin\theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \arcsin\frac{\pi}{2}$$

Just like the inverse functions, trigonometric inverse functions have the property that the range of the original function is its domain and vice versa.

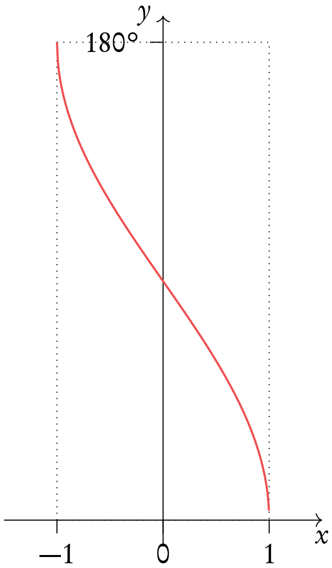
## $\sin^{-1} x = \arcsin x$



Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

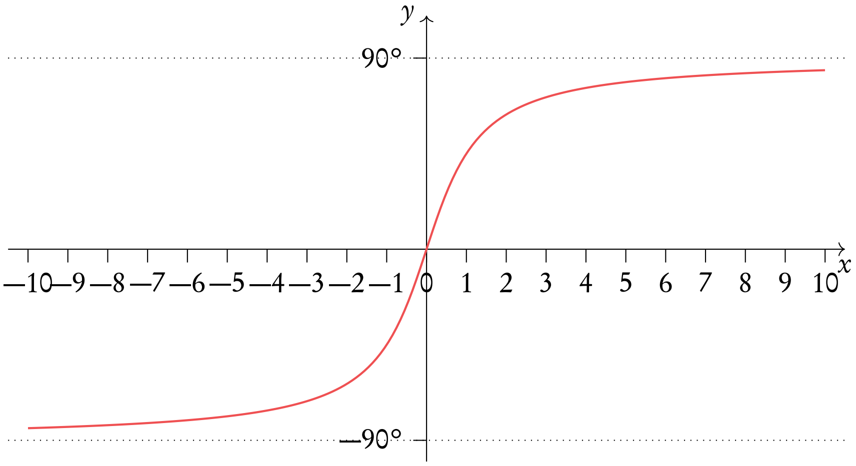
## $\cos^{-1} x = \arccos x$



Domain: $-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$

## $\tan^{-1} x = \arctan x$

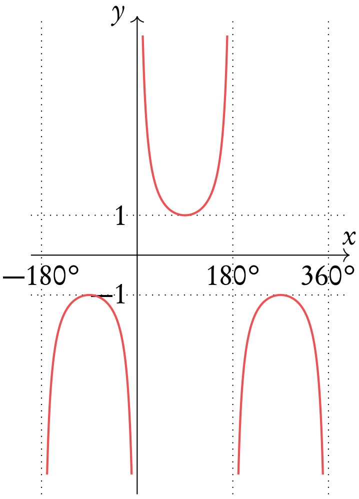


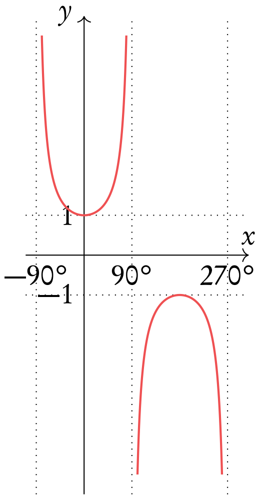
Domain: $x \in \mathbb{R}$

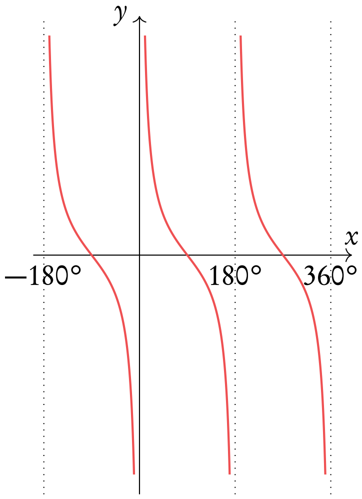
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

## Reciprocal trigonometric functions

$$\frac{1}{\sin\theta} = \csc\theta \qquad \frac{1}{\cos\theta} = \sec\theta \qquad \frac{1}{\tan\theta} = \cot\theta$$







These functions are the reciprocal functions, their vertical asyptotes correspond to the $x$-axis intercepts of the original function. The functions $\csc\theta$ and $\sec\theta$ are periodic with a period of 360°, $\cot\theta$ has a period of 180°.

5

# DIFFERENTIATION

## Table of contents & cheatsheet

### Definitions

**Differentiation** is a way to find the gradient of a function at any point, written as $f'(x)$, $y'$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

**Tangent line to a point on a curve** is a linear line with the same gradient as that point on the curve.

### 5.2. Polynomials 60

**Product** $y = uv$, then: $y' = uv' + u'v$

**Quotient** $y = \frac{u}{v}$, then: $y' = \frac{vu' - uv'}{v^2}$

**Chain** $y = g(u)$ where $u = f(x)$, then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

### 5.3. Tangent and normal 64

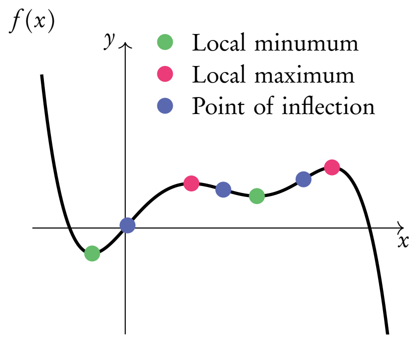
**Tangent** line with the same gradient as a point on a curve.

**Normal** perpendicular to the tangent $m = \frac{-1}{\text{slope of tangent}}$

Both are linear lines with general formula: $y = mx + c$.

1. Use derivative to find gradient of the tangent. For normal then do $-\frac{1}{\text{slope of tangent}}$.
2. Input the $x$-value of the point into $f(x)$ to find $y$.
3. Input $y$, $m$ and the $x$-value into $y = mx + c$ to find $c$.

### 5.4. Turning points 65



| | $f'(x)$ | $f''(x)$ |
|---|---|---|
| Local minimum | 0 | + |
| Local maximum | 0 | − |
| Points of inflection | | 0 |

### 5.5. Sketching graphs 69

Gather information before sketching:

**Intercepts** $x$-intercept: $f(x) = 0$
$y$-intercept: $f(0)$

**Turning points** minima: $f'(x) = 0$ and $f''(x) < 0$
maxima: $f'(x) = 0$ and $f''(x) > 0$
point of inflection: $f''(x) = 0$

**Asymptotes** vertical: $x$-value when the function divides by 0
horizontal: $y$-value when $x \to \infty$

Plug the found $x$-values into $f(x)$ to determine the $y$-values.

### 5.6. Applications 71

**Kinematics**

Derivative represents the rate of change, integration the reverse.

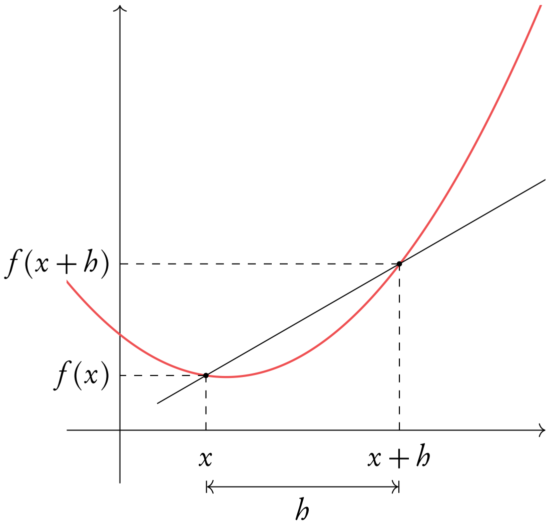
$$\frac{\mathrm{d}s}{\mathrm{d}t} = v \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = a$$

$$\begin{tikzcd} s \arrow[r, bend left] & v \arrow[r, bend left] & a \end{tikzcd}$$

displacement velocity acceleration

## 5.1 Derivation from first principles

As the derivative at a point is the gradient, differentiation can be compared to finding gradients of lines: $m = \frac{y_2 - y_1}{x_2 - x_1}$.



Using the graph

$x_1 = x$ $\quad x_2 = x + h$

$y_1 = f(x)$ $\quad y_2 = f(x+h)$

Plugging into the equation of the gradient of a line

$$m = \frac{f(x+h) - f(x)}{x + h - x}$$

Taking the limit of $h$ going to zero, such that the distance between the points becomes very small, one can approximate the gradient at a point of any funtion:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

## 5.2 Polynomials

As you have learnt in the section on functions, a straight line graph has a gradient. This gradient describes the rate at which the graph is changing and thanks to it we can tell how steep the line will be. In fact gradients can be found for any function - the special thing about linear functions is that their gradient is always the same (given by $m$ in $y = mx + c$). For polynomial functions the gradient is always changing. This is where calculus comes in handy; we can use differentiation to derive a function using which we can find the gradient for any value of $x$.

Using the following steps, you can find the derivative function ($f'(x)$) for any polynomial function ($f(x)$).

**Polynomial** a mathematical expression or function that contains several terms often raised to different powers

e.g. $y = 3x^2$, $y = 121x^5 + 7x^3 + x$ or $y = 4x^{\frac{2}{3}} + 2x^{\frac{1}{3}}$

**Principles** $y = f(x) = ax^n \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = nax^{n-1}$.

The (original) function is described by $y$ or $f(x)$, the derivative (gradient) function is described by $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$.

**Derivative of a constant (number)** 0

e.g. For $f(x) = 5$, $f'(x) = 0$

**Derivative of a sum** sum of derivatives.

If a function you are looking to differentiate is made up of several summed parts, find the derivatives for each part separately and then add them together again.

e.g. $f(x) = ax^n$ and $g(x) = bx^m$

$$f'(x) + g'(x) = nax^{n-1} + mbx^{m-1}$$

## 5.2.1 Rules

With more complicated functions, in which several functions are being multiplied or divided by one another (rather than just added or substracted), you will need to use the product or quotient rules.

DB 6.2

### Product rule

When functions are multiplied: $y = uv$

then: $y' = uv' + u'v$

which is the same as $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$.

e.g. $y = x^2 \cos x$, then $y' = x^2(\cos x)' + (x^2)' \cos x = -x^2 \sin x + 2x \cos x$

## Quotient rule

When functions are divided: $y = \dfrac{u}{v}$

then: $y' = \dfrac{vu' - uv'}{v^2}$

which is the same as $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.

e.g. $y = \dfrac{x^2}{\cos x}$, then $y' = \dfrac{(x^2)'\cos x - x^2(\cos x)'}{(\cos x)^2} = \dfrac{2x\cos x + x^2\sin x}{\cos^2 x}$

## Chain rule

Function inside another function: $y = g(u)$ where $u = f(x)$

then: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

### Differentiating with the chain rule.

Let $y = (\cos x)^2$, determine the derivative $y'$

| | | |
|---|---|---|
| **1.** | What is the outside function? What is the inside function? | Inside function: $u = \cos x$<br>Outside function: $y = u^2$ |
| **2.** | Find $u'$ and $y'$ | $u' = \dfrac{\mathrm{d}u}{\mathrm{d}x} = -\sin x;\quad y' = \dfrac{\mathrm{d}y}{\mathrm{d}u} = 2u$ |
| **3.** | Fill in formula | $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$<br>$= 2u(-\sin x)$<br>$= -2\sin x\cos x$ |

## 5.2.2 Implicit differentiation

When we have a function that does not express $y$ explicitely ($y =$) like in the previous methods, we must use *implicit differentiation*.

Steps to follow:

1. differentiate with respect to $x$, don't forget chain and $x$ rules. Derivative of $y$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$
2. collect/gather terms with $\frac{\mathrm{d}y}{\mathrm{d}x}$
3. solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$

**Implicit differentiation**

Find the gradient at point $(0,1)$ of $\mathrm{e}^{xy} + \ln\left(y^2\right) + \mathrm{e}^y = 1 + \mathrm{e}$

**1.** Treat each part seperately

$\mathrm{e}^{xy}$ becomes $y\mathrm{e}^{xy} + \frac{\mathrm{d}y}{\mathrm{d}x}x\mathrm{e}^{xy}$

$\ln\left(y^2\right)$ becomes $2y + \frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y}x\frac{\mathrm{d}y}{\mathrm{d}x}$

$\mathrm{e}^y$ becomes $\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^y$

**2.** Collect/gather terms with $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$y\mathrm{e}^{xy} + \frac{\mathrm{d}y}{\mathrm{d}x}x\mathrm{e}^{xy} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{2}{y} + \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^y = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(x\mathrm{e}^{xy} + \frac{2}{y} + \mathrm{e}^y\right) = -y\mathrm{e}^{xy}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y\mathrm{e}^{xy}}{x\mathrm{e}^{xy} + \frac{2}{y} + \mathrm{e}^y}$$

**3.** Solve for the point $(0,1)$

Substituting in $x = 0$ and $y = 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{2+\mathrm{e}}$$

## 5.3 Tangent and normal equation

**Tangent** a straight line that touches a curve at one single point. At that point, the gradient of the curve is equal to the gradient of the tangent.

**Normal** a straight line that is perpendicular to the tangent line:

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}}$$

For any questions with tangent and/or normal lines, use the steps described in the following example.

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

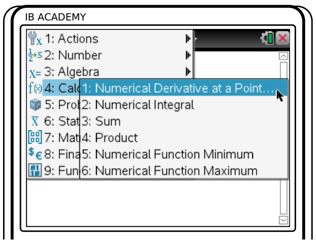
| **Finding the linear function of the tangent.** | | |
|---|---|---|
| | Let $f(x) = x^3$. Find the equation of the tangent at $x = 2$ | |
| **1.** | Find the derivative and fill in value of $x$ to determine slope of tangent | $f'(x) = 3x^2$<br>$f'(2) = 3 \cdot 2^2 = 12$ |
| **2.** | Determine the $y$ value | $f(x) = 2^3 = 8$ |
| **3.** | Plug the slope $m$ and the $y$ value in $y = mx + c$ | $8 = 12x + c$ |
| **4.** | Fill in the value for $x$ to find $c$ | $8 = 12(2) + c \Rightarrow c = -16$<br>eq. of tangent: $y = 12x - 16$ |

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

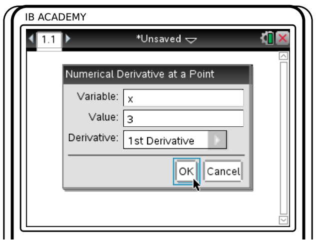
| **Finding the linear function of the normal.** | | |
|---|---|---|
| | Let $f(x) = x^3$. Find the equation of the normal at $x = 2$ | |
| **1.** | | $f'(2) = 12$ |
| **2.** | | $f(x) = 8$ |
| **3.** | Determine the slope of the normal $m = \dfrac{-1}{\text{slope tangent}}$ and plug it and the $y$-value into $y = mx + c$ | $m = \dfrac{-1}{12}$<br>$8 = -\dfrac{1}{12}x + c$ |
| **4.** | Fill in the value for $x$ to find $c$ | $8 = -\dfrac{1}{12}(2) + c \Rightarrow c = \dfrac{49}{6}$<br>eq. of normal: $y = -\dfrac{1}{12}x + \dfrac{49}{6}$ |

**To find the gradient of a function for any value of $x$.**

$f(x) = 5x^3 - 2x^2 + x$. Find the gradient of $f(x)$ at $x = 3$.



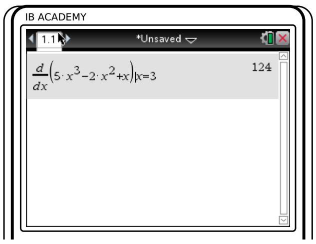
Press menu
`4: Calculus`
`1: Numerical Derivative at a Point`



Variable: x
Value: 3
Derivative: 1st Derivative

Enter the variable used in your function (x) and the value of $x$ that you want to find. Keep the settings on `1st Derivative`

Press OK



$\frac{d}{dx}(5\cdot x^3-2\cdot x^2+x)|x=3$ 124

Type in your function press enter

In this case, $f'(3) = 124$

## 5.4 Turning points

There are three types of turning points:

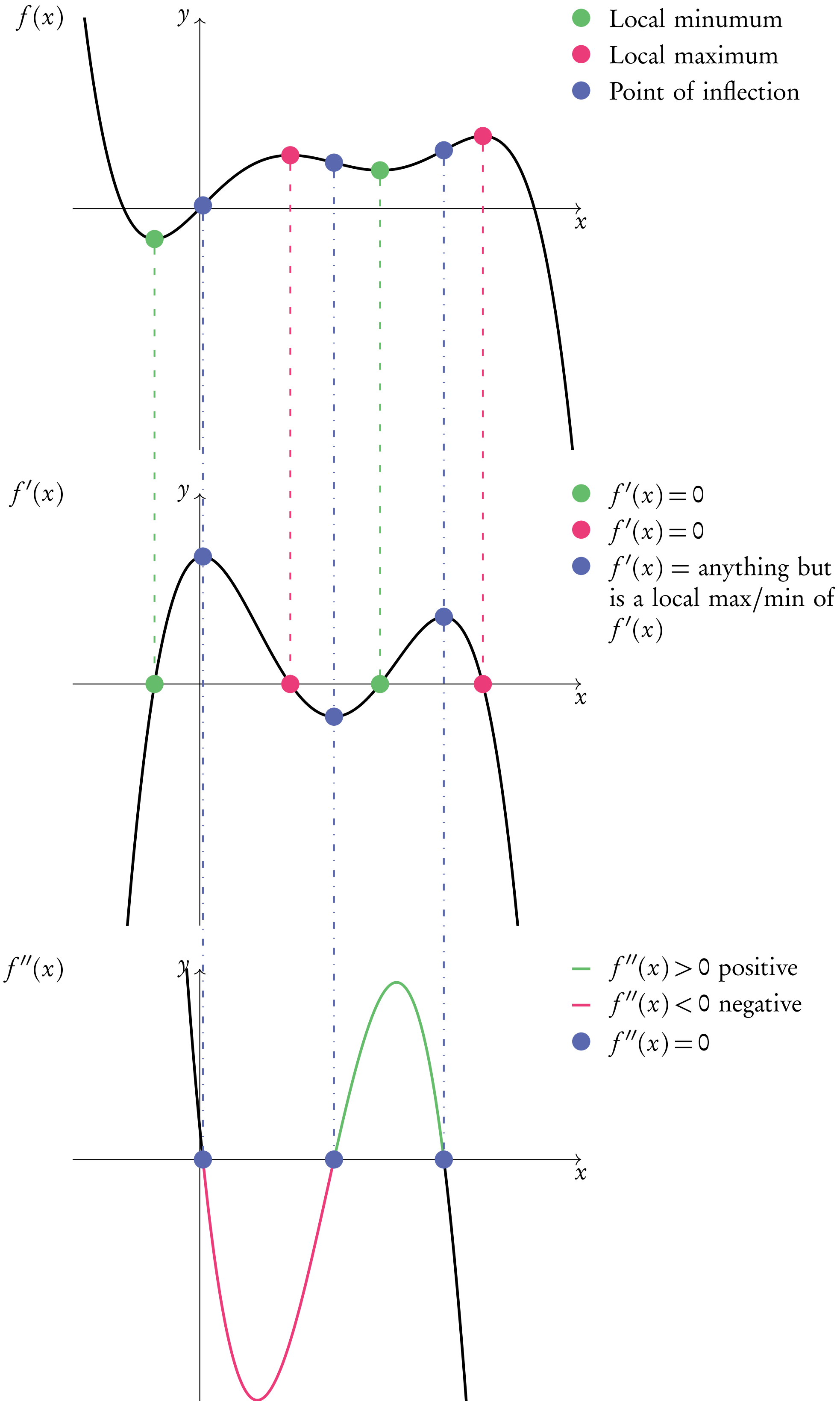
1. Local maxima
2. Local minima
3. Points of inflection

We know that when $f'(x) = 0$ there will be a maximum or a minimum. Whether it is a maximum or minimum should be evident from looking at the graph of the original function. If a graph is not available, we can find out by plugging in a slightly smaller and slightly larger value than the point in question into $f'(x)$. If the smaller value is negative and the larger value positive then it is a local minimum. If the smaller value is positive and the larger value negative then it is a local minimum.

If you take the derivative of a derivative function (one you have already derived) you get the *second derivate*. In mathematical notation, the second derivative is written as $y''$, $f''(x)$ or $\frac{d^2y}{dx^2}$. We can use this to determine whether a point on a graph is a maximum, a minimum or a point of inflection as demonstrated in the following Figure 5.1.

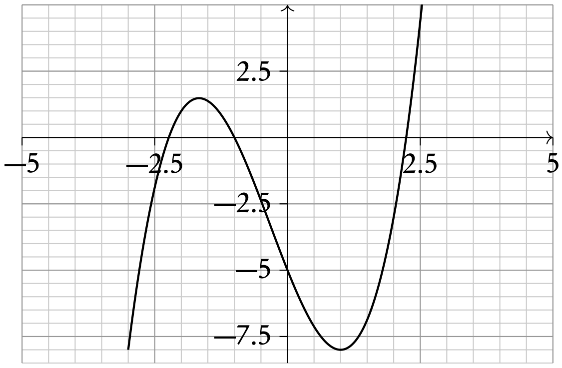
Notice how the points of inflection of $f(x)$ are minima and maxima in $f'(x)$ and thus equal 0 in $f''(x)$

Figure 5.1: Graph that shows a local maximum, a local minimum and points of inflection



## Finding turning points.

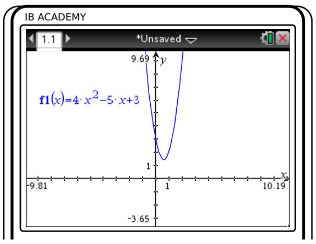
The function $f(x) = x^3 + x^2 - 5x - 5$ is shown. Use the first and second derivative to find turning points: the minima, maxima and points of inflection (POI).



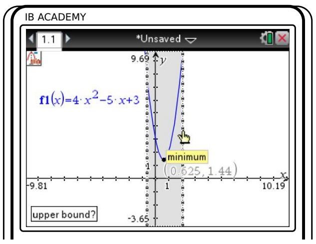
| | | |
|---|---|---|
| **1.** | Find the first and second derivative. | $f'(x) = 3x^2 + 2x - 5$<br>$f''(x) = 6x + 2$ |
| **2.** | Find $x_{\text{min}}$ and $x_{\text{max}}$ by setting $f'(x) = 0$. | $3x^2 + 2x - 5 = 0$<br>GDC yields: $x = 1$ or $x = -\frac{5}{3}$ |
| **3.** | Find $y$-coordinates by inserting the $x$-value(s) into the original $f(x)$. | $f(1) = (1)^3 + (1)^2 - 5(1) - 5 = -8$,<br>so $x_{\text{min}}$ at $(1, -8)$.<br>$f\left(-\frac{5}{3}\right) = \left(-\frac{5}{3}\right)^3 + \left(-\frac{5}{3}\right)^2$<br>$-5\left(-\frac{5}{3}\right) - 5 = 1.48(3\text{ s.f.})$,<br>so $x_{\text{max}}$ at $\left(-\frac{5}{3}, 1.48\right)$. |
| **4.** | Find POI by setting $f''(x) = 0$ | $6x + 2 = 0$ |
| **5.** | Then enter values of $x$ into original function to find coordinates | $f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 - 5\left(-\frac{1}{3}\right) - 5$<br>$y = -3.26$ (3 s.f.)<br>so POI at $\left(-\frac{1}{3}, -3.26\right)$ |

## To find turning points (local maximum/minimum) of a function

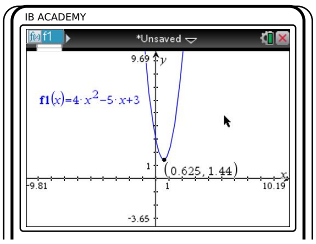
Find the coordinates of the local minimum for $f(x) = 4x^2 - 5x + 3$



Press menu
`6: Analyze graph`
or `2: Minimum`
or `3: Maximum`



Use the cursor to set the bounds (the min/max must be between the bounds)



So the coordinates of the minimum for $f(x)$ are $(0.625, 1.44)$

## 5.5 Sketching graphs

When sketching a graph, you will need the following information:

1. Intercepts,
2. Turning points (maximums, minimums and inflection points) and
3. Asymptotes

### Sketching a function.

Sketch the function $f(x) = \dfrac{x^2}{x^2-16}$

**1.** Note down all information:

1. Intercepts:
   - $y$-intercept: $f(0)$
   - $x$-intercept: $f(x) = 0$
2. Turning points:
   - min/max: $f'(x) = 0$
   - inflection: $f''(x) = 0$
3. Asymptotes:
   - vertical: denominator = 0, $x = -b$, for $\log(x+b)$
   - horizontal: $\lim\limits_{x\to\infty \text{ or } x\to-\infty}$, $y = c$, for $a^x + c$

To find the $y$-coordinate, input the $x$-value into the original $f(x)$.

1. $y$-intercept when $x = 0$:

   $$f(0) = \frac{0^2}{0^2-16} = 0 \quad (0,0)$$

   $$f(x) = \frac{x^2}{x^2-16} = 0 \quad x = 0 \quad \text{(same)}$$

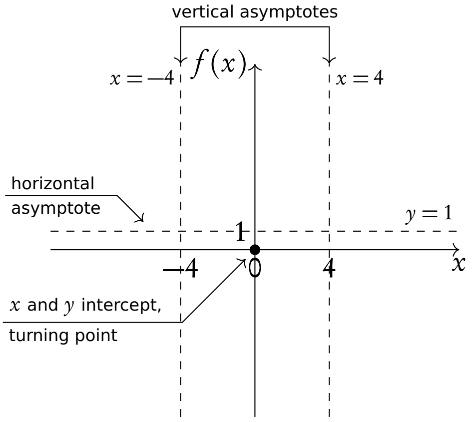
   This is the only $x$-intercept.
2. Turning point: $f'(x) = \dfrac{-32x}{x^2-16^2}$, $x = 0$ $(0,0)$ (Found with quotient rule). $f' = 0$ when $x = 0$.
3. Vertical asymptotes when $x^2 - 16 = 0$, so $x = 4$ and $x = -4$.

   Horizontal asymptote:

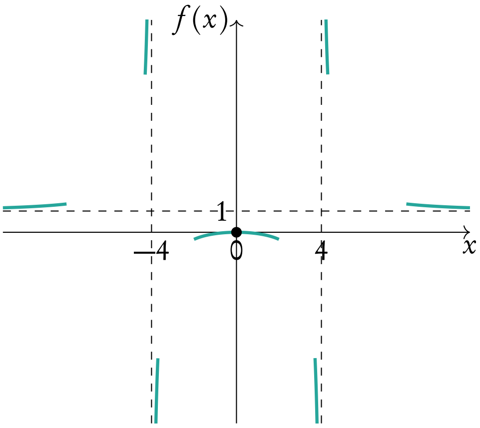
   $$\lim_{x\to\infty} f(x) = \frac{x^2}{x^2} = 1, \text{ so } y = 1$$

**2.** Mark out information on axis

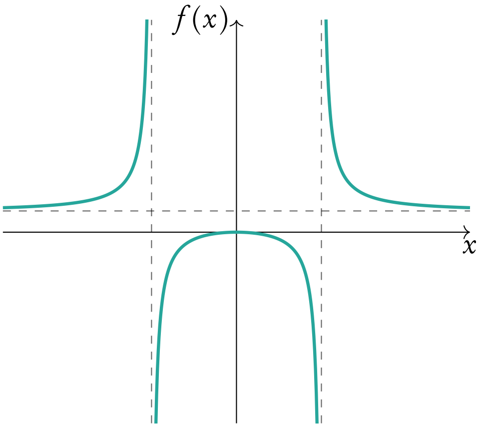
Clearly indicate them to guarantee marks



**3.** Think about where your lines are coming from



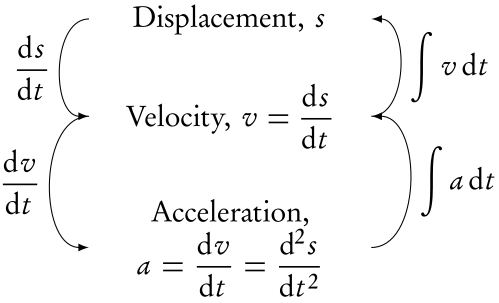
**4.** Join the dots



# 5.6 Applications

## 5.6.1 Kinematics

Kinematics deals with the movement of bodies over time. When you are given one function to calculate displacement, velocity or acceleration you can use differentiation or integration to determine the functions for the other two.



The derivative represents the rate of change, i.e. the gradient of a graph. So, velocity is the rate of change in displacement and acceleration is the rate of change in velocity.

**Answering kinematics questions.**

A diver jumps from a platform at time $t = 0$ seconds. The distance of the diver above water level at time $t$ is given by $s(t) = -4.9t^2 + 4.9t + 10$, where $s$ is in metres. Find when velocity equals zero. Hence find the maximum height of the diver.

| | | |
|---|---|---|
| **1.** | Find an equation for velocity by differentiating equation for distance | $v(t) = -9.8t + 4.9$ |
| **2.** | Solve for $v(t) = 0$ | $-9.8t + 4.9 = 0, \quad t = 0.5$ |
| **3.** | Put value into equation for distance to find height above water | $s(0.5) = -4.9(0.5)^2 + 4.9(0.5) + 10 = 11.225\,\text{m}$ |

### 5.6.2 Optimization

We can use differentiation to find minimum and maximum areas/volumes of various shapes. Often the key skill with these questions is to find an expression using simple geometric formulas and rearranging in order to differentiate.

**Finding the min/max area or volume**

The sum of height and base of a triangle is $40\,\text{cm}$. Find an expression for its area in terms of $x$, its base length. Hence find its maximum area.

| | | |
|---|---|---|
| **1.** | Find expressions for relevant dimensions of the shape | length of the base $(b) = x$<br>$\text{height} + \text{base} = 40$<br>so $h + x = 40$<br>area of triangle $A = \frac{1}{2}xh$ |
| **2.** | Reduce the number of variables by solving the simultaneous equations | Since $h = 40 - x$, substitute $h$ into $A$:<br>$A = \frac{1}{2}x(40 - x) = -\frac{1}{2}x^2 + 20x$ |
| **3.** | Differentiate | $f'(x) = -x + 20$ |
| **4.** | Find $x$ when $f'(x) = 0$ | $-x + 20 = 0 \Rightarrow x = 20$ |
| **5.** | Plug $x$ value in $f(x)$ | $-\frac{1}{2}20^2 + 20(20) = -200 + 400 = 200\,\text{cm}^2$ |

If an expression is given in the problem, skip to step 2 (e.g. cost/profit problems).

# 5.7 Implicit differentiation

The derivative of $y$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$! Differentiate with respect to $x$ always.

**Find the gradient at point $(0,1)$ of $\mathrm{e}^{xy}+\ln y^2+\mathrm{e}^y=1+\mathrm{e}$**

| | | |
|---|---|---|
| **1.** | Differentiate with respect to $x$ | $\mathrm{e}^{xy}\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{1}{y^2}\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)+$<br>$\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ |
| **2.** | Gather terms with $\frac{\mathrm{d}y}{\mathrm{d}x}$ | $\frac{\mathrm{d}y}{\mathrm{d}x}\left(x\mathrm{e}^{xy}+\frac{2}{y}+\mathrm{e}^y\right)+y\mathrm{e}^{xy}=0$ |
| **3.** | Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ | $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-y\mathrm{e}^{xy}}{x\mathrm{e}^{xy}+\frac{2}{y}+\mathrm{e}^y}$ at (0,1)<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{2+\mathrm{e}}$ |

# INTEGRATION

6

## Table of contents & cheatsheet

### 6.1. Indefinite integral 76

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

**Integration with an internal function**

$$\int f(ax+b)\,\mathrm{d}x$$

Integrate normally and multiply by $\frac{1}{\text{coefficient of } x}$

**Integration by substitution**

$$\int f\left(g(x)\right)\cdot g'(x)\,\mathrm{d}x$$

### 6.2. Definite integral 78

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,\mathrm{d}x$$

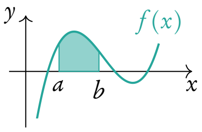
Be careful, the order you substitute $a$ and $b$ into the indefinite integral is relevant for your answer:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

**Area between a curve and the $x$-axis**

By determining a definite integral for a function, you can find the area beneath the curve that is between the two $x$-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x$$

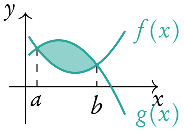


Note: the area below the $x$-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the $x$-axis. Sketching the graph will show what part of the function lies below the $x$-axis.

**Area between two curves**

Using definite integrals you can also find the areas enclosed between curves.

$$A_{\text{between}} = \int_a^b \left(g(x) - f(x)\right)\mathrm{d}x$$



With $g(x)$ as the "top" function (furthest from the $x$-axis). For the area between curves, it does not matter what is above/below the $x$-axis.

**Volume of revolution**

$$V = \pi \int_a^b y^2 \,\mathrm{d}x = \int_a^b \pi y^2 \,\mathrm{d}x$$

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis - this is called the volume of revolution.

## 6.1 Indefinite integral and boundary condition

Integration is essentially the opposite of derivation. The following equation shows how to integrate a function:

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \qquad\qquad n \neq -1$$

As you can see, every time you integrate the power on your variable will increase by 1 (this is opposite of what happens when you derive, then it always decreases). Whenever you integrate you also **always add** $+C$ to this function. This accounts for any constant that may have been lost while deriving. As you may have noticed, whenever you do derivation any constants that were in the original function, $f(x)$, become 0 in the derivative function, $f'(x)$. In order to determine the value of $C$, you need to fill in a point that lies on the curve to set up an equation in which you can solve for $C$. (Note: this is the same thing you need to do when finding the $y$-intercept, $C$, for a linear function – see Functions: Linear functions).

**Standard integration.**

Let $f'(x) = 12x^2 - 2$
Given that $f(-1) = 1$, find $f(x)$.

| | | |
|---|---|---|
| **1.** | Separate summed parts (optional) | $\int 12x^2 - 2\,\mathrm{d}x = \int 12x^2\,\mathrm{d}x + \int -2\,\mathrm{d}x$ |
| **2.** | Integrate | $f(x) = \int 12x^2\,\mathrm{d}x + \int -2\,\mathrm{d}x =$<br>$\frac{12}{3}x^3 - 2x + C$ |
| **3.** | Fill in values of $x$ and $f(x)$ to find $C$ | Since $f(-1) = 1$,<br>$4(-1)^3 - 2(-1) + C = 1$<br>$C = 3$<br>So: $f(x) = 4x^3 - 2x + 3$ |

## 6.1.1 Integration with an internal function

$$\int f(ax+b)\,\mathrm{d}x \qquad \text{integrate normally and multiply by } \frac{1}{\text{coefficient of } x}$$

**Example.**

Find the following integrals:

$$\int \mathrm{e}^{3x-4}\,\mathrm{d}x$$

Coefficient of $x = 3$, so

$\int \mathrm{e}^{3x-4}\,\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x-4} + C$

$$\int \cos(5x-2)\,\mathrm{d}x$$

Coefficient of $x = 5$, so

$\int \cos(5x-2)\,\mathrm{d}x = \frac{1}{5}\sin(5x-2) + C$

## 6.1.2 Integration by substitution

$$\int f\left(g(x)\right)\cdot g'(x)\,\mathrm{d}x$$

Integration by substitution: usually these questions will be the most complicated-looking integrals you will have to solve. So if an integration question looks complicated, try to look for a function and its derivative inside the function you are looking to integrate; it is likely to be a question where you have to use the substitution method! Study the example to see how it's done.

**Integrate by substitution**

| | | |
|---|---|---|
| | Find $\int 3x^2 \mathrm{e}^{x^3}\,\mathrm{d}x$ | |
| **1.** | Identify the inside function $u$, this is the function whose derivative is also inside $f(x)$. | $g(x) = u = x^3$ |
| **2.** | Find the derivative $u' = \frac{\mathrm{d}u}{\mathrm{d}x}$ | $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ |
| **3.** | Substitute $u$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$ into the integral<br>(this way $\mathrm{d}x$ cancels out) | $\int \mathrm{e}^u \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \mathrm{e}^u\,\mathrm{d}u = \mathrm{e}^u + C$ |
| **4.** | Substitute $u$ back to get a function with $x$ | $\int \mathrm{e}^u + C = \mathrm{e}^{x^3} + C$ |

## 6.2 Definite integral

If there are limit values indicated on your integral, you are looking to find a definite integral. This means that these values will be used to find a numeric answer rather than a function.

This is done in the following way, where the values for $a$ and $b$ are substituted as $x$-values into your indefinite integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,\mathrm{d}x$$

Be careful, the order you substitute $a$ and $b$ into the indefinite integral is relevant for your answer:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

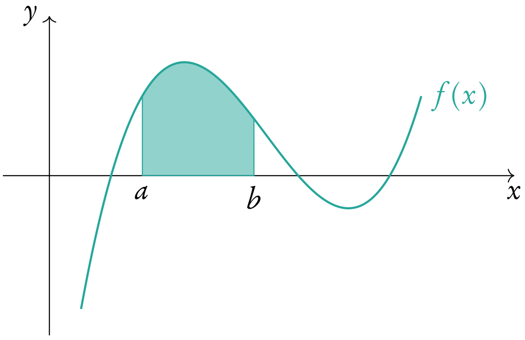
**Solving definite integrals.**

Find $\int_3^7 12x^2 - 2\,\mathrm{d}x$, knowing that $F(x) = 4x^3 - 2x$

| | | |
|---|---|---|
| **1.** | Find the indefinite integral<br>(without $+C$) | $\int_3^7 12x^2 - 2\,\mathrm{d}x = \left[4x^3 - 2x\right]_3^7$ |
| **2.** | Fill in: $F(b) - F(a)$<br>(integral $x = b$) $-$ (integral $x = a$) | $= \left[4(7)^3 - 2(7)\right] - \left[4(3)^3 - 2(3)\right]$<br>$= 1256$ |

## 6.2.1 Area

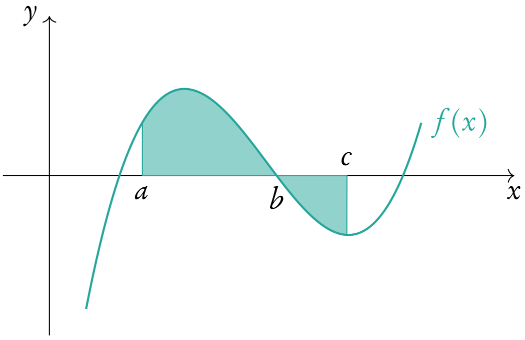
### Area between a curve and the $x$-axis



DB 6.5

By determining a definite integral for a function, you can find the area beneath the curve that is between the two $x$-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x$$



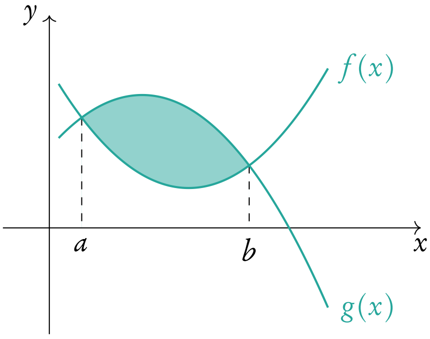
Note: the area below the $x$-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the $x$-axis. Sketching the graph will show what part of the function lies below the $x$-axis. So

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x + \left|\int_b^c f(x)\,\mathrm{d}x\right|$$

or

$$A_{\text{curve}} = \int_a^c |f(x)|\,\mathrm{d}x$$

### Area between two curves



Using definite integrals you can also find the areas enclosed between curves:

$$A_{\text{between}} = \int_a^b \big(g(x) - f(x)\big)\,\mathrm{d}x$$

With $g(x)$ as the “top” function (furthest from the $x$-axis). For the area between curves, it does not matter what is above/below the $x$-axis.

## Finding areas with definite integrals.

Let $y = x^3 - 4x^2 + 3x$

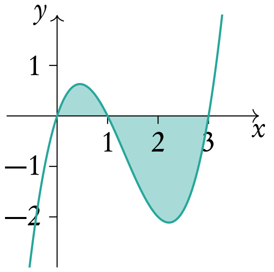
Find the area from $x = 0$ to $x = 3$.

**1.** Find the $x$-intercepts: $f(x) = 0$

$x^3 - 4x^2 + 3x = 0$, using the GDC:

$x = 0$ or $x = 1$ or $x = 3$

**2.** If any of the $x$-intercepts lie within the range, sketch the function to see which parts lie above and below the $x$-axis.



**3.** Setup integrals and integrate

Left: $\displaystyle\int_0^1 x^3 - 4x^2 + 3x \, \mathrm{d}x =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1$$

$$= \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0)$$

$$= \frac{5}{12}$$

Right: $\displaystyle\int_1^3 x^3 - 4x^2 + 3x \, \mathrm{d}x =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$$

$$= \left(\frac{1}{4}(3)^4 - \frac{4}{3}(3)^3 + \frac{3}{2}(3)^2\right)$$

$$- \left(\frac{1}{4}(1)^4 - \frac{4}{3}(1)^3 + \frac{3}{2}(1)^2\right)$$

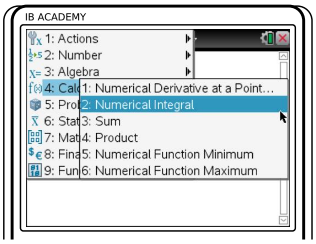
$$= -\frac{8}{3}$$

**4.** Add up the areas (and remember areas are never negative!)

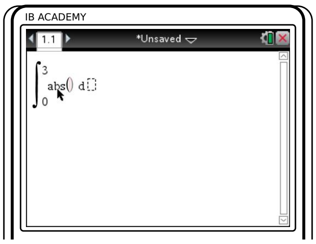
$$\frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

## Alternatively, use the calculator to find areas

Calculate the area between $\int_0^3 x^3 - 4x^2 + 3x$ and the $x$-axis



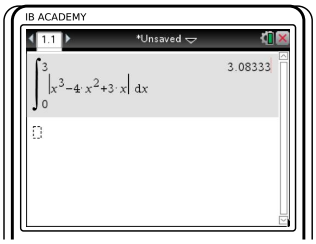
Press menu
`4: Calculus`
`2: Numerical integral`



$\int_0^3 \text{abs}()\,d$

Enter the boundaries and before putting the function.

Press 📖, choose ‘`abs(`’



$\int_0^3 |x^3-4\cdot x^2+3\cdot x|\,dx$ 3.08333

Enter the function and place the variable (usually $x$) after $d$

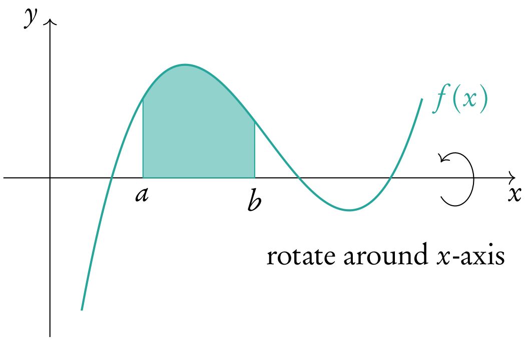
In this case, the area is 3.083

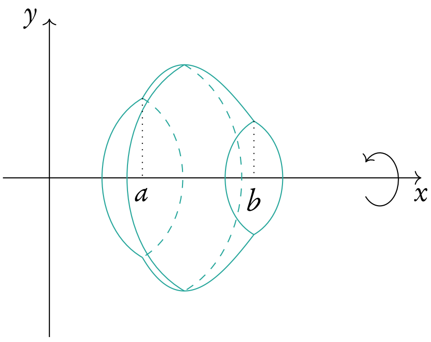
## 6.2.2 Volume of revolution

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis — this is called the volume of revolution.

DB 6.5

$$V = \pi \int_a^b y^2\,dx \quad \equiv \quad V = \int_a^b \pi y^2\,dx$$





Example.

Find the area from $x = 1$ to $x = 4$ for the function $y = \sqrt{x}$.

$$A = \int_1^4 \sqrt{x}\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \left[\frac{2}{3}(4)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}}\right] = \frac{14}{3}$$

This area is rotated 360° ($= 2\pi$) around the $x$-axis. Find the volume of the solid.

$$V = \pi\int_1^4 \sqrt{x}^2\,\mathrm{d}x = \pi\int_1^4 x\,\mathrm{d}x = \pi\left[\frac{1}{2}x^2\right]_1^4 = \pi\left(\left[\frac{1}{2}(4)^2\right] - \left[\frac{1}{2}(1)^2\right]\right) = \frac{15\pi}{2}$$

### 6.2.3 Integration by parts

DB

General statement:

$$\int u\cdot\mathrm{d}v = uv - \int v\cdot\mathrm{d}u \qquad \text{or} \qquad \int f(x)g'(x)\,\mathrm{d}x = fg - \int f'g\,\mathrm{d}x$$

Example.

Solve $\int x\sin x\,\mathrm{d}x$

If $f(x) = x$ then $f'(x) = 1$ and the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$

$$-x\cos x - \int 1\cdot\sin x\,\mathrm{d}x = -x\cos x + \cos x + C$$

Sometimes it may be necessary to do intergration by parts multiple times.

Example.

Solve $\int \mathrm{e}^{2x}\sin x\,\mathrm{d}x$

We know that $\sin x$ is the derivative of $\cos x$

$$u = \sin x \qquad \mathrm{d}u = \cos x\,\mathrm{d}x$$

$$\mathrm{d}v = \mathrm{e}^{2x}\,\mathrm{d}x \qquad v = \frac{\mathrm{e}^{2x}}{2}$$

Using the formula given in the data booklet and the information above:

$$\int \mathrm{e}^{2x}\sin x\,\mathrm{d}x = \frac{\mathrm{e}^{2x}\sin x}{2} - \int \frac{\mathrm{e}^{2x}}{2}\cos x\,\mathrm{d}x$$

Unfortunately this is still not nice to solve so we need to repeat the procedure along the same line of reasoning for the integral $\int \frac{\mathrm{e}^{2x}}{2}\cos x\,\mathrm{d}x$.

We know trigonometric functions recur – taking the integral twice would bring us back to the same trigonometric identity, apart from containing the opposite sign.

**Example.**

$$u = \cos x \qquad du = -\sin x\,dx$$

$$dv = \frac{e^{2x}}{2}\,dx \qquad v = \frac{e^{2x}}{4}$$

We again construct the formula of integration by parts.

$$\int \frac{e^{2x}}{2}\cos x\,dx = \cos x\frac{e^{2x}}{4} - \int \frac{e^{2x}}{4}\sin x\,dx$$

Now we want to combine both equations to solve for the original integral.

$$\int e^{2x}\sin x\,dx = \frac{e^{2x}\sin x}{2} - \cos x\frac{e^{2x}}{4} + \int \frac{e^{2x}}{4}\sin x\,dx$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{e^{2x}\sin x}{2} - \cos x\frac{e^{2x}}{4}$$

$$= \frac{2}{5}\sin x e^{2x} - \frac{1}{5}\cos x e^{2x}$$

# 7

# PROBABILITY

## Table of contents & cheatsheet

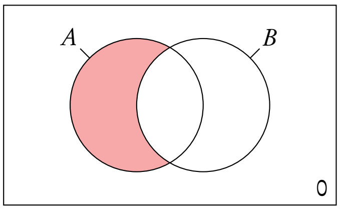
### Definitions

**Sample space** the list of all possible outcomes.

**Event** the outcomes that meet the requirement.

**Probability** for event $A$, $P(A) = \dfrac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$.

### 7.1. Single events 86



### 7.2. Multiple events 88

Probabilities for successive events can be expressed through tree diagrams.
In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

**Conditional probability** used for successive events that come one after another (as in tree diagrams).

The probability of $A$, given that $B$ has happened: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

### 7.3. Distributions 90

**Expected values**

$$E(X) = \sum xP(X = x)$$

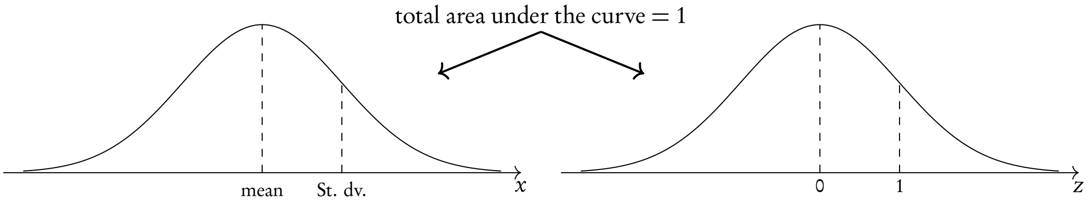
**Binomial distribution with parameters $n$ and $p$** it should be used for situations with only 2 outcomes and lots of trials

$$P(X = x) = \binom{n}{r} p^r (1-p)^{n-r}$$

where $\binom{n}{r} = \text{nCr} = \dfrac{n!}{r!(n-r)!}$, $n =$ number of trials, $p =$ probability of success, $r =$ number of success.

On calculator:

- Binompdf(n,p,r) $P(X = r)$
- Binomcdf(n,p,r) $P(x \leq r)$
- Mean np
- Varience npq

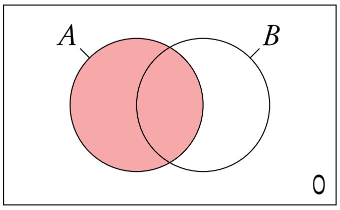


Notation: $X \sim N(\mu, \sigma^2)$ Transform to standardised normal distribution: $Z = \dfrac{x - \mu}{\sigma}$

On calculator: normal cdf (lower bound, upper bound, mean $(= \mu)$, standard deviation $(= \sigma)$)

## 7.1 Single events (Venn diagrams)

Probability for single events can be expressed through venn diagrams.



**Sample space** the list of all possible outcomes.

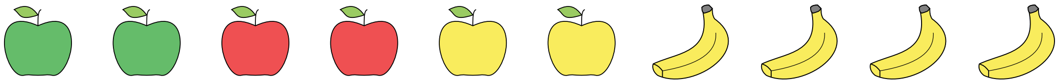
**Event** the outcomes that meet the requirement.

**Probability** for event $A$,

$$P(A) = \frac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$$

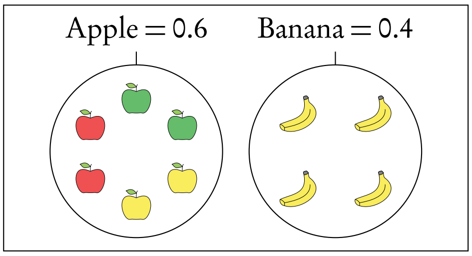
Here the shaded circle.

Imagine I have a fruit bowl containing 6 apples and 4 bananas.



**Example.**

I pick a piece of fruit.
*What is the probability of picking each fruit?*

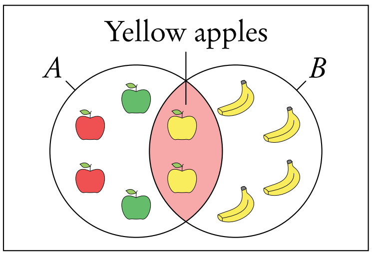


In independent events $P(A\cap B) = P(A)\times P(B)$. It will often be stated in questions if events are independent.

As apples cannot be bananas this is mutually exclusive, therefore $P(A\cup B) = P(A) + P(B)$ and $P(A\cap B) = 0$. It is also an exhaustive event as there is no other options apart from apples and bananas. If I bought some oranges the same diagram would then be not exhaustive (oranges will lie in the Sample Space).

**Example.**

Of the apples 2 are red, 2 are green and 2 are yellow.
*What is the probability of picking a yellow apple?*



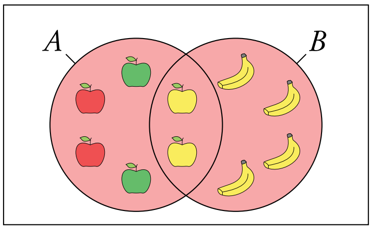
A: apples

B: yellow fruit

This is not mutually exclusive as both apples and bananas are yellow fruits. Here we are interested in the intersect $P(A \cap B)$ of apples and yellow fruit, as a yellow apple is in both sets $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

**Example.**

*What is the probability of picking an apple or a yellow fruit?*



A: apples

B: yellow fruit

This is a union of two sets: apple and yellow fruit.

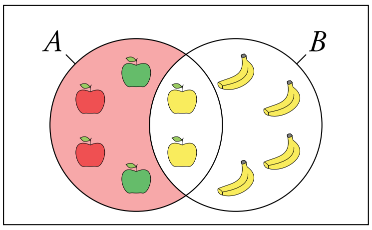
When an event is exhaustive the probability of the union is 1.

The union of events $A$ and $B$ is:

- when $A$ happens;
- when $B$ happens;
- when both $A$ and $B$ happen $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

**Example.**

*What is the probability of not picking a yellow fruit?*



A: apples

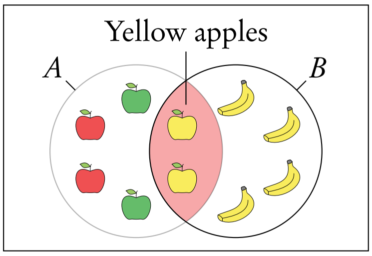
B: yellow fruit

This is known as the compliment of $B$ or $B'$. $B' = 1 - B$.

Here we are interested in everything but the yellow fruit.

**Example.**

*What is the probability of picking an apple given I pick a yellow fruit?*

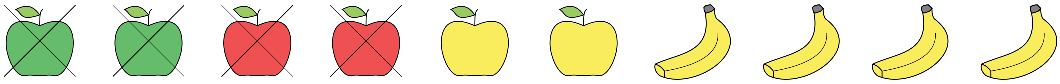


A: apples

B: yellow fruit

This is "conditional" probability in a single event. **Do not use the formula in the formula booklet.** Here we are effectively narrowing the sample space $= \frac{0.2}{(0.2+0.4)} = \frac{1}{3}$.

You can think of it like removing the non yellow apples from the fruit bowl before choosing.



Conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

## 7.2 Multiple events (tree Diagrams)

Questions involving dependent events will often involve elements that are drawn "without replacement". Remember that the probabilities will be changing with each new set of branches.

**Dependent events** two events are dependent if the outcome of event $A$ affects the outcome of event $B$ so that the probability is changed.

**Independent events** two events are independent if the fact that $A$ occurs does not affect the probability of $B$ occurring.

**Conditional probability** used for successive events that come one after another (as in tree diagrams). The probability of $A$, given that $B$ has happened: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

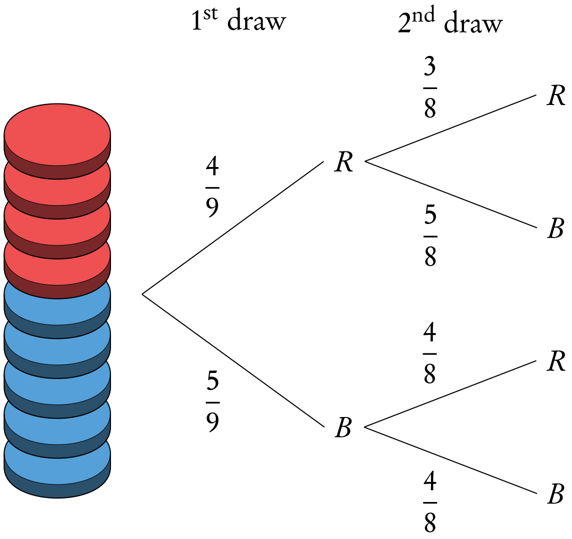
Probabilities for successive events can be expressed through tree diagrams. In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

**Example.**

### Two disks are randomly drawn without replacement from a stack of 4 red and 5 blue disks. Draw a tree diagram for all outcomes.

The probability of drawing two red disks can be found by multiplying both probabilities of getting red $\left(\frac{4}{9} \times \frac{3}{8}\right)$.



$R$ and $R$: $\frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$

$R$ and $B$: $\frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

$B$ and $R$: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

$B$ and $B$: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

The probabilities for each event should always add up to 1. The probabilities describing all the possible outcomes should also equal 1 (that is, the probabilities that we found by multiplying along the individual branches).

### What is the probability to draw one red and one blue disk?

$P$(one red **and** one blue)

$(P(R) \text{ and } P(B))$ **or** $(P(B) \text{ and } P(R))$

$$(P(R) \times P(B)) \qquad (P(B) \times P(R))$$

$$\frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

It is common for conditional probability questions to relate to previous answers.

### What is the probability to draw at least one red disk?

$P$(**at least** one red)

$$P(R \text{ and } R) + P(B \text{ and } R) + P(R \text{ and } B) = 1 - P(B \text{ and } B)$$

$$\frac{12}{72} + \frac{20}{72} + \frac{20}{72} = 1 - \frac{20}{72} = \frac{52}{72} = \frac{13}{18}$$

### What is the probability of picking a blue disc given that at least one red disk is picked?

$$P(\text{blue disk} \mid \text{at least one red disk}) = \frac{P(\text{a blue disk})}{P(\text{at least one red disk})} = \frac{\frac{5}{9}}{\frac{13}{18}} = \frac{10}{13}$$

Another way of dealing with multiple events is with a sample space diagram or a probability distribution.

**Probability distributions.**

A fair coin is tossed twice, $X$ is the number of heads obtained.

**1.** Draw a sample space diagram

|   | H | T |
|---|---|---|
| H | H, H | H, T |
| T | T, H | T, T |

**2.** Tabulate the probability distribution

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X=x)$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

(The sum of $P(X=x)$ always equals 1)

**3.** Find the expected value of $X$: $E(X)$

$$E(X) = \sum xP(X=x)$$
$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

So if you toss a coin twice, you expect to get heads once.

## 7.3 Distributions

**Probability distribution** a list of each possible value and their respective probabilities.

### 7.3.1 Distribution by function

A probability distribution can also be given by a function.

**Probability distribution by function.**

$P(X=x) = k\left(\frac{1}{3}\right)^{x-1}$ for $x = 1, 2, 3$. Find constant $k$.

**1.** Use the fact that $\sum P(X=x) = 1$

$$k\left(\frac{1}{3}\right)^{1-1} + k\left(\frac{1}{3}\right)^{2-1} + k\left(\frac{1}{3}\right)^{3-1} = 1$$

**2.** Simplify and solve for $k$

$k + \frac{1}{3}k + \frac{1}{9}k = \frac{13}{9}k = 1$. So, $k = \frac{9}{13}$.

## 7.3.2 Binomial distribution

**Binomial distribution with parameters $n$ and $p$** it should be used for situations with only 2 outcomes and lots of trials

$$P(X=x)=\binom{n}{r}p^{r}(1-p)^{n-r}$$

where $\binom{n}{r}=\text{nCr}=\frac{n!}{r!(n-r)!}$

$n=$ number of trials

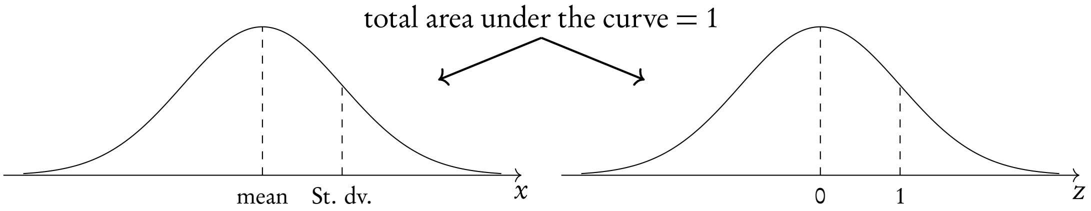
$p=$ probability of success

$r=$ number of success

## 7.3.3 Normal distribution

A normal distribution is one type of probability distribution which gives a bell-shape curve if all the values and their corresponding probabilities are plotted.

We can use normal distributions to find the probability of obtaining a certain value or a range of values. This can be found using the area under the curve; the area under the bell-curve between two $x$-values always corresponds to the probability for getting an $x$-value in this range. The total area under the normal distribution is always 1; this is because the total probability of getting any $x$-value adds up to 1 (or, in other words, you are 100% certain that your $x$-value will lie somewhere on the $x$-axis below the bell-curve).



Even though you will be using your GDC to find probabilities for normal distributions, it's always *very* useful to draw a diagram to indicate for yourself (and the examiner) what area or $x$-value you are looking for.

**Notation:** $X \sim N(\mu,\sigma^2)$

**Transform to standard N:** $Z=\frac{x-\mu}{\sigma}$

**On calculator:** `normal cdf` (lower bound, upper bound, mean $(=\mu)$, standard deviation $(=\sigma)$)

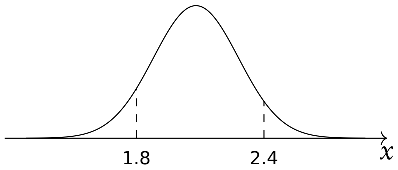
### Finding mean and standard deviation of a normal distribution.

All nails longer than $2.4\,\mathrm{cm}$ (5.5%) and shorter than $1.8\,\mathrm{cm}$ (8%) are rejected. What is the mean and standard deviation length?

**1.** Write down equations

$$P(L < 1.8) = 0.08$$
$$P(L > 2.4) = 0.055$$

**2.** Draw a sketch!



**3.** Write standardized equation of the form $P(Z < \ldots)$

$$P\left(Z < \frac{1.8 - \mu}{\sigma}\right) = 0.08$$
$$P\left(Z > \frac{2.4 - \mu}{\sigma}\right) = 0.055$$
$$P\left(Z < \frac{2.4 - \mu}{\sigma}\right) = 1 - 0.055 = 0.945$$

**4.** Use "`inVnorm`" on calculator

$$\texttt{inVnorm}\,(0.08, 0, 1) = -1.4051$$
$$\texttt{inVnorm}\,(0.945, 0, 1) = 1.5982$$

**5.** Equate and solve

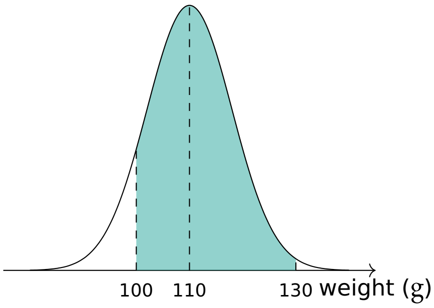
$$\begin{cases} \dfrac{1.8 - \mu}{\sigma} = -1.4051 \\ \dfrac{2.4 - \mu}{\sigma} = 1.5982 \end{cases}$$
$$\begin{cases} \mu = 2.08 \\ \sigma = 0.200 \end{cases}$$

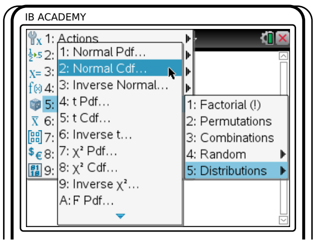
## To find a probability or percentage of a whole (the area under a normal distribution curve)

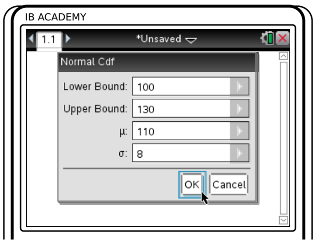
The weights of pears are normally distributed with mean = 110 g and standard deviation = 8 g.
Find the percentage of pears that weigh between 100 g and 130 g

Sketch!
Indicate:

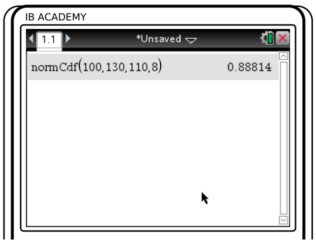
- The mean = 110 g
- Lower bound = 100 g
- Upper bound = 130 g
- And shade the area you are looking to find.







Lower Bound: 100
Upper Bound: 130
μ: 110
σ: 8



```
normCdf(100,130,110,8)                0.88814
```

Press menu, choose
`5: Probability`
`5: Distributions`
`2: Normal Cdf`

Enter lower and upper boundaries, mean ($\mu$) and standard deviation ($\sigma$).
For lower bound $= -\infty$,
set `lower: -1E99`
For upper bound $= \infty$,
set `upper: 1E99`

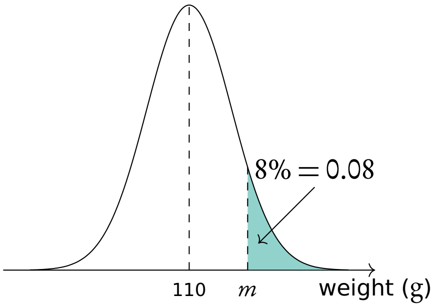
Press OK

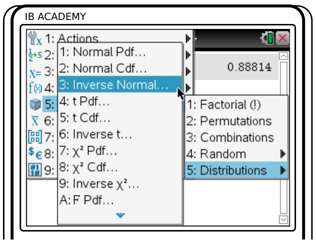
So 88.8% of the pears weigh between 100 g and 130 g.

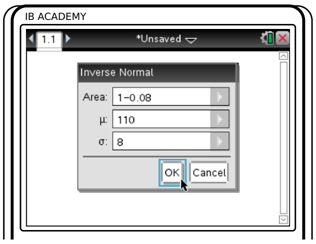
## To find an $x$-value when the probability is given

The weights of pears are normally distributed with mean $= 110$ g and standard deviation $= 8$ g. 8% of the pears weigh more than $m$ grams. Find $m$.

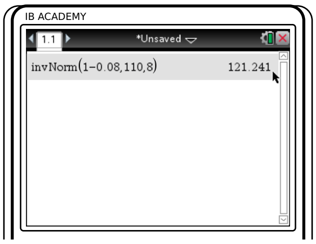
Sketch!







Area: 1−0.08
μ: 110
σ: 8



```
invNorm(1−0.08,110,8)        121.241
```

Press menu
`5: Probability`
`5: Distributions`
`3: Inverse Normal`

Enter probability (`Area`), mean ($\mu$) and standard deviation ($\sigma$).
The calculator assumes the area is to the left of the $x$-value you are looking for.
So in this case:
area $= 1 - 0.08 = 0.92$

Press OK

So $m = 121$, which means that 8% of the pears weigh more than 121 g.

## 7.4 Poisson distribution

Used to calculate the probabilities of various numbers of "successes". Each "success" must be independent. i.e. if mean number of calls to a fire station on weekday is 8. What is probability that on a given weekday there would be 11 calls?

$$p = \frac{e^{-u} u^x}{x!}$$

where:

$u =$ mean "successes"
$x =$ number of "successes" in question

### 7.4.1 Bayes theorem

| | |
|---|---|
| **Bayes theorem** | the probability that $B$ is true given that $A$ is true |

$$P(B|A) = \frac{P(A)P(A|B)}{P(B)}$$

# STATISTICS

8

## Table of contents & cheatsheet

### Definitions

**Population** the entire group from which statistical data is drawn (and which the statistics obtained represent).

**Sample** the observations actually selected from the population for a statistical test.

**Random Sample** a sample that is selected from the population with no bias or criteria; the observations are made at random.

**Discrete** finite or countable number of possible values (e.g. money, number of people)

**Continuous** infinite amount of increments (e.g. time, weight)

Note: continuous data can be presented as discrete data, e.g. if you round time to the nearest minute or weight to the nearest kilogram.

### 8.1. Descriptive statistics 98

**Mean** the average value,

$$\bar{x} = \frac{\text{the sum of the data}}{\text{no. of data points}}$$

**Mode** the value that occurs most often

**Median** when the data set is ordered low to high and the number of data points is:

- odd, then the median is the middle value;
- even, then the median is the average of the two middle values.

**Range** largest $x$-value − smallest $x$-value

**Variance** $\sigma^2 = \frac{\sum f(x-\bar{x}^2)}{n}$ *calculator only*

**Standard deviation** $\sigma = \sqrt{\text{variance}}$ *calculator only*

**Grouped data:** data presented as an interval. Use the midpoint as the $x$-value in all calculations.

$Q_1$ first quartile = 25th percentile.

$Q_2$ median = 50th percentile

$Q_3$ third quartile = 75th percentile

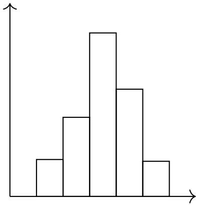
$Q_3 - Q_1$ interquartile range (IQR) = middle 50 percent

### 8.2. Statistical graphs 100

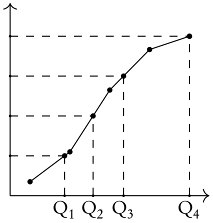
**Frequency** the number of times an event occurs in an experiment

**Cumulative frequency** the sum of the frequency for a particular class and the frequencies for all the classes below it

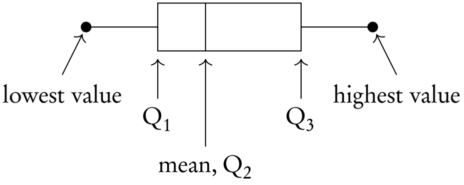
**Histogram**



**Cumulative frequency**



**Box and whisker plot**



### 8.3. Bi-variate analysis 104

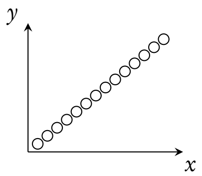
**Interpretation of $r$-values**

| $r$-value | correlation |
|---|---|
| $0.00 \le \lvert r \rvert \le 0.25$ | very weak |
| $0.25 \le \lvert r \rvert \le 0.50$ | weak |
| $0.50 \le \lvert r \rvert \le 0.75$ | moderate |
| $0.75 \le \lvert r \rvert \le 1.00$ | strong |

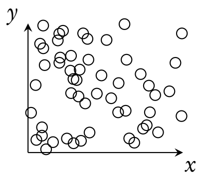
*Correlation does not mean causation.*

**Scatter diagrams**

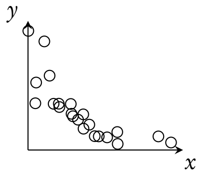
Perfect positive



No correlation



Weak negative



## 8.1 Descriptive statistics

The mean, mode and median, are all ways of measuring "averages". Depending on the distribution of the data, the values for the mean, mode and median can differ slightly or a lot. Therefore, the mean, mode and median are all useful for understanding your data set.

**Example data set:** 6, 3, 6, 13, 7, 7 in a table:

| | | | | |
|---|---|---|---|---|
| $x$ | 3 | 6 | 7 | 13 |
| **frequency** | 1 | 2 | 2 | 1 |

**Mean** the average value, $\bar{x} = \dfrac{\text{the sum of the data}}{\text{no. of data points}} = \dfrac{\sum x}{n} = \dfrac{\sum f x}{\sum f}$

**Mode** the value that occurs most often (highest frequency)

**Median** the middle value when the data set is ordered low to high. Even number of values: the median is the average of the two middle values. Find for larger values as $n + \frac{1}{2}$.

**Range** largest $x$-value − smallest $x$-value

**Variance** $\sigma^2 = \dfrac{\sum f(x - \bar{x}^2)}{n}$ *calculator only*

**Standard deviation** $\sigma = \sqrt{\text{variance}}$ *calculator only*

**Note on grouped data:** data presented as an interval; e.g. 10–20 cm.

- Use the midpoint as the $x$-value in all calculations. So for 10–20 cm use 15 cm.
- For 10–20 cm, 10 is the lower boundary, 20 is the upper boundary and the width is $20 - 10 = 10$.

Adding a constant to all the values in a data set or multiplying the entire data set by a constant influences the mean and standard deviation values in the following way:

Table 8.1: Adding or multiplying by a constant

| | **adding constant $k$** | **multiplying by $k$** |
|---|---|---|
| **mean** | $\bar{x} + k$ | $k \times \bar{x}$ |
| **standard deviation** | $\sigma$ | $k \times \sigma$ |

| | | |
|---|---|---|
| $Q_1$ | first quartile | $=$ 25th percentile.<br>The value for $x$ so that 25% of all the data values are $\leq$ to it. |
| $Q_2$ | median | $=$ 50th percentile |
| $Q_3$ | third quartile | $=$ 75th percentile |
| $Q_3 - Q_1$ | interquartile range (IQR) | $=$ middle 50 percent |

**Example.**

**Snow depth is measured in centimeters:**
**$30, 75, 125, 55, 60, 75, 65, 65, 45, 120, 70, 110$.**
**Find the range, the median, the lower quartile, the upper quartile and the interquartile range.**

First always rearrange data into ascending order: $30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125$

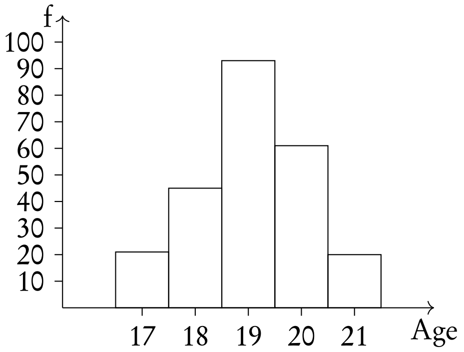
1. The range:
$$125 - 30 = 95\,\text{cm}$$
2. The median: there are 12 values so the median is between the 6th and 7th value.
$$\frac{65+70}{2} = 67.5\,\text{cm}$$
3. The lower quartile: there are 12 values so the lower quartile is between the 3rd and 4th value.
$$\frac{55+60}{2} = 57.5\,\text{cm}$$
4. The upper quartile: there are 12 values so the lower quartile is between the 9th and 10th value.
$$\frac{75+110}{2} = 92.5\,\text{cm}$$
5. The IQR
$$92.5 - 57.5 = 35\,\text{cm}$$

## 8.2 Statistical graphs

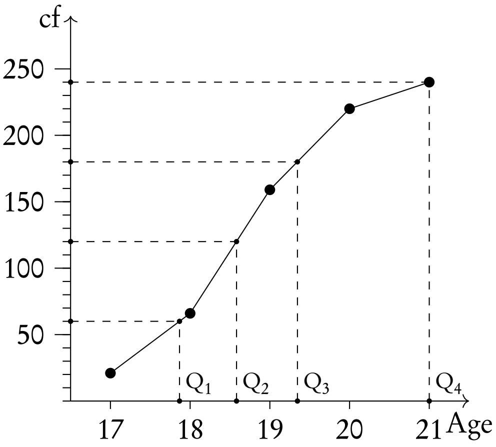
**Frequency** the number of times an event occurs in an experiment

**Cumulative frequency** the sum of the frequency for a particular class and the frequencies for all the classes below it

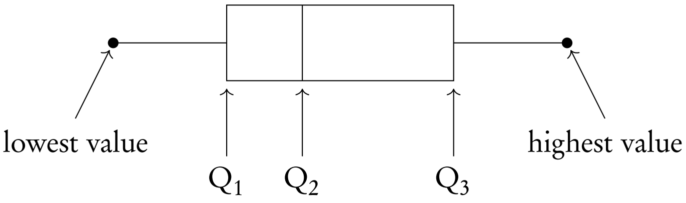
| Age | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|
| No. of students | 21 | 45 | 93 | 61 | 20 |
| Cumulative freq. | 21 | 66 | 159 | 220 | 240 |



A histogram is used to display the frequency for a specific condition. The frequencies (here: # of students) are displayed on the $y$-axis, and the different classes of the sample (here: age) are displayed on the $x$-axis. As such, the differences in frequency between the different classes assumed in the sample can easily be compared.



The cumulative frequency graph is used to display the development of the frequencies as the classes of the event increase. The graph is plotted by using the sum of all frequencies for a particular class, added to the frequencies for all the classes below it. The classes of the event (age) are displayed on the $x$-axis, and the frequency is displayed on the $y$-axis. The cumulative frequency graph always goes upwards, because the cumulative frequency increases as you include more classes.



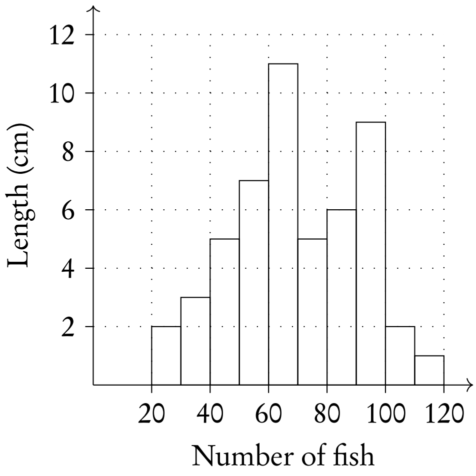
Box and whisker plots neatly summarize the distribution of the data. It gives information about the range, the median and the quartiles of the data. The first and third quartiles are at the ends of the box, the median is indicated with a vertical line in the interior of the box, and the maximum and minimum points are at the ends of the whiskers.

**Outliers** will be any points lower than $Q_1 - 1.5 \times \text{IQR}$ and larger than $Q_3 + 1.5 \times \text{IQR}$ (IQR = interquartile range)

To identify the value of $Q_1$, $Q_2$ and $Q_3$, it is easiest to use the cumulative frequency graph. First, determine the percentage of the quartile in question. Second, divide the total cumulative frequency of the graph (i.e. the total sample size) by 100 and multiply by the corresponding percentage. Then, you will have found the frequency ($y$-value) at which 25% for $Q_1$ / 50% for $Q_2$ / 75% for $Q_3$ of the sample is represented. To find the $x$-value, find the corresponding $x$-value for the previously identified $y$-value.

**Example.**

**Using the histogram, create a cumulative frequency graph and use it to construct a box and whisker diagram.**

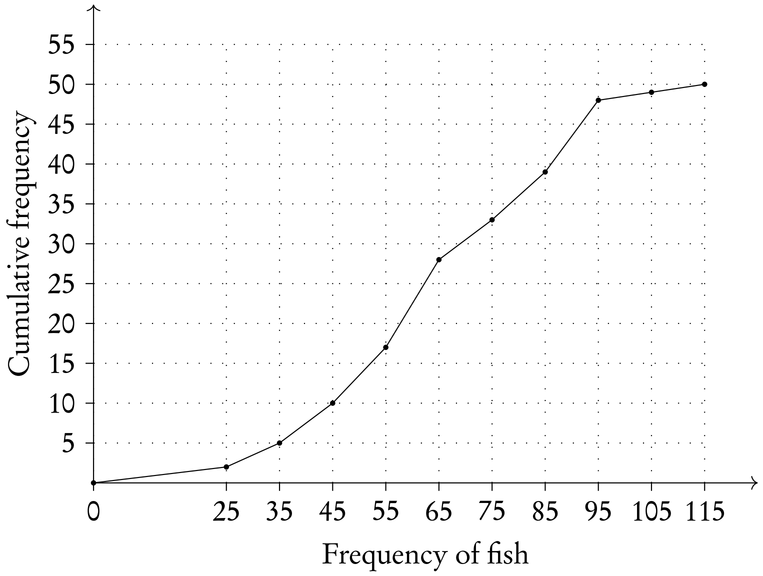


Write out the table for frequency and cumulative frequency.

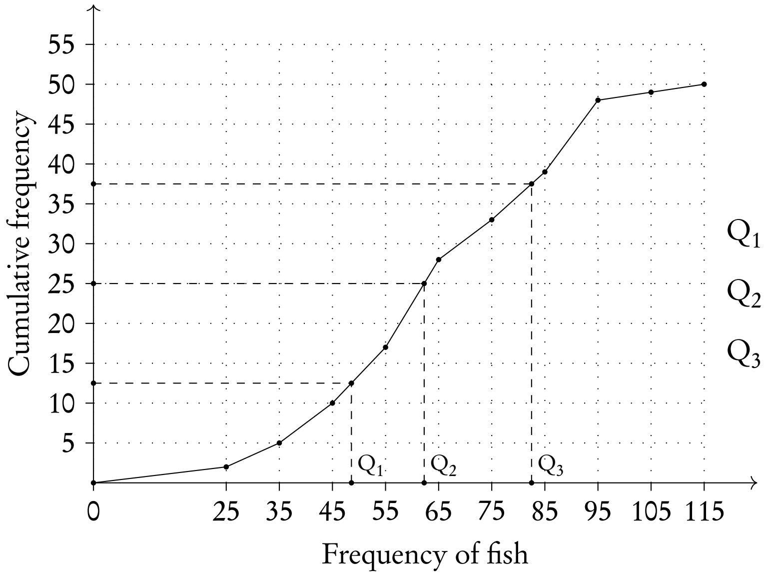
| **Frequency of fish** | 20–30 | 30–40 | 40–50 | 50–60 | 60–70 | 70–80 | 80–90 | 90–100 | 100–110 | 110–120 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Length of fish** | 2 | 3 | 5 | 7 | 11 | 5 | 6 | 9 | 1 | 1 |
| **Cumulative f.** | 2 | 5 | 10 | 17 | 28 | 33 | 39 | 48 | 49 | 50 |

**Example.**

Plot on cumulative frequency chart. Remember to use the midpoint of the date, e.g., 25 for 20–30.



Use graph to find $Q_1$, $Q_2$ and $Q_3$.

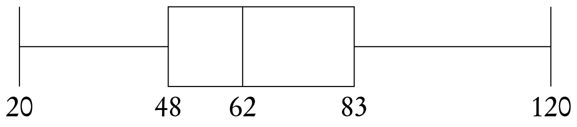


$Q_1$ 25% of $50 = 12.5$ $\rightarrow 48$

$Q_2$ 50% of $50 = 25$ $\rightarrow 62$

$Q_3$ 75% of $50 = 37.5$ $\rightarrow 83$

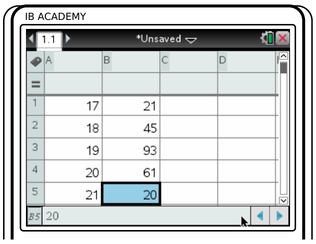
Plot box and whiskers.



## GDC

### To find mean, standard deviation and quartiles etc.

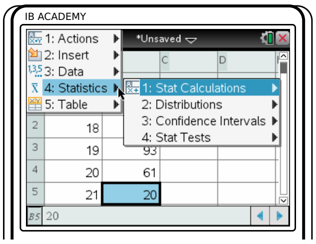
For the data used in the previous example showing the ages of students



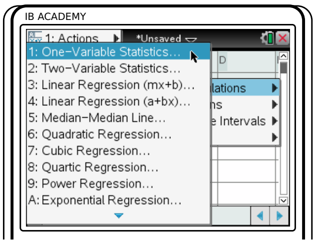
| A | B |
|---|---|
| 17 | 21 |
| 18 | 45 |
| 19 | 93 |
| 20 | 61 |
| 21 | 20 |

B5 20

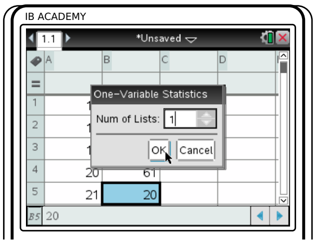
Press on, go to `Lists and Spreadsheets`. Enter $x$-values in `L1` and, if applicable, frequencies in `L2`



Press menu, choose `4: Statistics` `1: Stat Calculations`



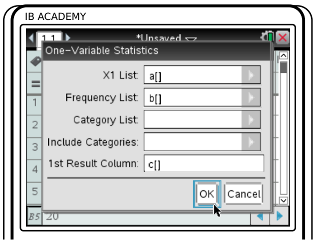
`1: One-Variable Statistics`



Num of Lists: 1

Enter `Num of lists: 1`.

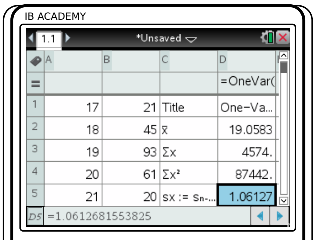
Press OK



X1 List: a[]
Frequency List: b[]
Category List:
Include Categories:
1st Result Column: c[]

Enter names of columns you used to enter your $x$-list and frequency list and column where you would like the solutions to appear: `a[]`, `b[]` and `c[]`.

Press OK



| A | B | C | D |
|---|---|---|---|
| | | | =OneVar( |
| 17 | 21 | Title | One–Va… |
| 18 | 45 | $\bar{x}$ | 19.0583 |
| 19 | 93 | Σx | 4574. |
| 20 | 61 | Σx² | 87442. |
| 21 | 20 | sx := sn-… | 1.06127 |

D5 =1.0612681553825

mean = 19.06;
standard deviation = 1.06
etc.

## 8.3 Bi-variate analysis

Bi-variate analysis is a method of assessing how two (bi) sets of data (variables) correlate to one another. We use Pearson's correlation to put a number to this relationship

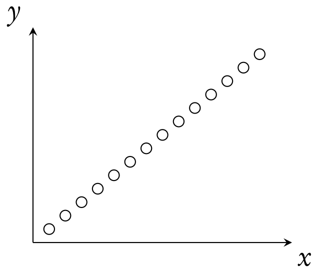
**Pearson's correlation $r$** is a measure to assess the linear correlation between two variables, where 1 is total positive correlation, 0 is no correlation, and −1 is total negative correlation.

Interpretation of $r$-values:

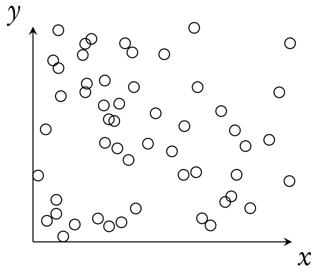
| $r$-value | $0 \leq \lvert r \rvert \leq 0.25$ | $0.25 \leq \lvert r \rvert \leq 0.50$ | $0.50 \leq \lvert r \rvert \leq 0.75$ | $0.75 \leq \lvert r \rvert \leq 1$ |
|---|---|---|---|---|
| correlation | very weak | weak | moderate | strong |

### Scatter diagrams

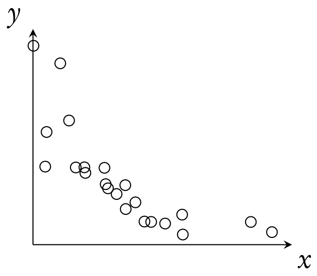
Perfect positive correlation $r = 1$



No correlation $r = 0$



Weak negative correlation
$-0.5 < r < -0.25$



However it is important to remember this maxim: *Correlation does not mean causation*.

Just because two variables have a relationship it does not mean they cause one another. For example Ice cream sales show a strong correlation to the number deaths by drowning. Therefore we might falsely state ice cream consumption causes drowning. But it is more plausible that both are caused by warm weather leading to more desire for ice cream and swimming and are just correlated.

## Using GDC

Calculate by finding the regression equation on your GDC: make sure `STAT DIAGNOSTICS` is turned `ON` (can be found when pressing `MODE`).

Bivariate statistics can also be used to predict a mathematical model that would best describe the relationship between the two measured variables; this is called regression. Here you will only have to focus on linear relationships, so only straight line graphs and equations.

Your ‘comment’ on Pearson’s correlation always has to include two things:

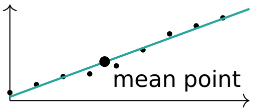
1. Positive / negative *and*
2. Stong / moderate / weak / very weak

### Find Pearson’s correlation $r$ and comment on it

The height of a plant was measured the first 8 weeks

| Week $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| Height (cm) $y$ | 23.5 | 25 | 26.5 | 27 | 28.5 | 31.5 | 34.5 | 36 | 37.5 |

**1.** Plot a scatter diagram



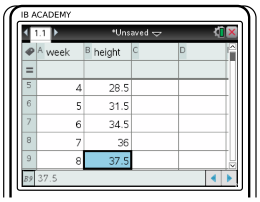
**2.** Use the mean point to draw a best fit line

$$\bar{x} = \frac{0+1+2+\ldots+8}{9} = 3.56$$

$$\bar{y} = \frac{23.5+25+\ldots+37.5}{9} = 30$$

**3.** Find the equation of the regression line Using GDC

$y = 1.83x + 22.7$



IB ACADEMY

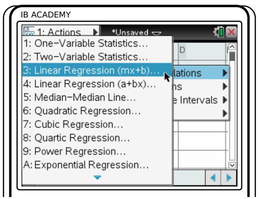
1.1 *Unsaved

| | A week | B height | C | D |
|---|---|---|---|---|
| = | | | | |
| 5 | 4 | 28.5 | | |
| 6 | 5 | 31.5 | | |
| 7 | 6 | 34.5 | | |
| 8 | 7 | 36 | | |
| 9 | 8 | 37.5 | | |

B9 37.5

Press on, got to “`Lists and Spreadsheets`”

Enter $x$-values in one column (e.g A) and $y$-values in another column (e.g. B)



IB ACADEMY

1: Actions *Unsaved

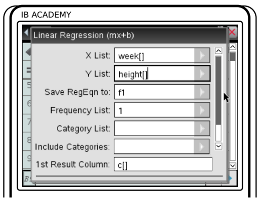
1: One–Variable Statistics…
2: Two–Variable Statistics…
3: Linear Regression (mx+b)…
4: Linear Regression (a+bx)…
5: Median–Median Line…
6: Quadratic Regression…
7: Cubic Regression…
8: Quartic Regression…
9: Power Regression…
A: Exponential Regression…

Press menu

`4: Statistics`

`1: Stat Calculations`

`3: Linear Regression (mx+b)`



IB ACADEMY

Linear Regression (mx+b)

X List: week[]
Y List: height[]
Save RegEqn to: f1
Frequency List: 1
Category List:
Include Categories:
1st Result Column: c[]

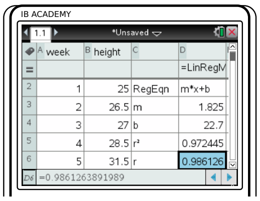
Enter

`X list: A [];`

`Y list: B[];`

`1st Result Column: C[]`

Press OK



IB ACADEMY

1.1 *Unsaved

| | A week | B height | C | D |
|---|---|---|---|---|
| = | | | | =LinRegM |
| 2 | 1 | 25 | RegEqn | m*x+b |
| 3 | 2 | 26.5 | m | 1.825 |
| 4 | 3 | 27 | b | 22.7 |
| 5 | 4 | 28.5 | r² | 0.972445 |
| 6 | 5 | 31.5 | r | 0.986126 |

D6 =0.9861263891989

So, equation of regression line is $y = 1.83x + 22.7$ and Pearson’s correlation ($r$-value) $= 0.986$

**4.** Comment on the result.

Pearson’s correlation is $r = 0.986$, which is a strong positive correlation.